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Operating Characteristics of Crosscorrelator With or Without Sample Mean Removal

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Preface

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The characteristic function of the output of a crosscorrelator, with the sample means removed from each channel, is derived in closed form. More generally, if scaled versions of the sample means are subtracted prior to multiplication of the channel inputs and summation, a closed form for the characteristic function of the correlator output is derived. These results are used to plot the exact operating characteristics of the crosscorrelator, as functions of the threshold, the general scaling factors applied to the sample means, the number

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of terms, N, summed to yield the output, the actual means at the inputs, and the signal-to-noise ratios of the random signal components at each of the system inputs. Programs for the various cases considered are documented and exercised. Comparisons are made with a Gaussian approximation, which can be used to extend the results to larger values of N than considered here, if needed. Asymptotic results for the exceedance distribution functions have also been derived, but they are not too useful for large N.

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LIST OF SYMBOLS

N	Number of terms summed to yield crosscorrelator output
u_n, v_n	Two channel inputs at sample time n
overbar	Statistical average
μ_u, μ_v	Means in two input channels
σ_u, σ_v	Standard deviations in two input channels
ρ	Correlation coefficient between two input channels
$u_s(n)$, $v_s(n)$	Random signal components at inputs
$u_d(n)$, $v_d(n)$	Random noise disturbances at inputs
S_u, S_v	Powers of random signal components
ρ_s	Correlation coefficient between signal components
D_u, D_v	Powers of random noise disturbances
R_u, R_v	Signal-to-noise ratios of random components of two input channels, (8)
$\tilde{u}_n$, $\tilde{v}_n$	Sample ac components of inputs
q	Crosscorrelator output, (12)
α, β	Scale factors applied to sample means
γ	Scale factor utilized in crosscorrelator output, (12)
U, V	Column matrices of two channel inputs
Q	NxN matrix, (16)
1	Column matrix of ones, (17)
$p_q(u)$	Probability density function of random variable q and argument u
$f_q(\xi)$	Characteristic function of random variable q and argument ξ
det	Determinant
$E_1, E_2, F_1, F_2, G_1, G_2$	Parameters of characteristic function, (24)
$\chi_q(n)$	n-th cumulant of random variable q
μ_q, σ_q	Mean and standard deviation of q
$I_\nu(z)$	Modified Bessel function of first kind
$K_\nu(z)$	Modified Bessel function of second kind
$P_q(u)$	Cumulative distribution function of q
$1-P_q(u)$	Exceedance distribution function of q
h	Normalized crosscorrelator output, (49), (94)
R	Signal-to-noise ratio for identical signal components, (56A), (78)

LIST OF SYMBOLS (Cont'd)

C_+, C_-	Contours for determining cumulative and exceedance distribution functions, (63), (64)
P_F, P_D	False alarm and detection probabilities
ϕ	Normalized Gaussian probability density function, (84)
Φ	Normalized Gaussian cumulative distribution function, (84)
Φ_I	Inverse Φ function
m_k, σ_k	Mean and standard deviation under hypothesis k, (86), (128)
r_u, r_v	Normalized means in two channels, (97)
r	Common normalized mean, (99), (124)
ω	Auxiliary variable equal to 1+2R, (102)
$U(u)$	Unit step function, (109)
$Q(a,b)$	Q-function of Marcum
$Q_M(a,b)$	Q_M-function, ref. 8
$q_M(u),\tilde{q}_M(u)$	Auxiliary functions, (115)

OPERATING CHARACTERISTICS OF CROSSCORRELATOR WITH OR WITHOUT SAMPLE MEAN REMOVAL

INTRODUCTION

The detection of weak signals in two channels is often accomplished by crosscorrelating the two waveforms and comparing with a threshold. For the case where a large number of independent products are added to yield the correlator output, the central limit theorem is often employed, with questionable validity for low false alarm probabilities, i.e. large thresholds. Also, this approximation may not be valid for intermediate numbers of terms added.

Here we wish to get exact operating characteristics for the crosscorrelator, namely detection probability vs. false alarm probability, even for probabilities as low as 1E-10. In particular, we desire results for an arbitrary number of products summed, for any degree of correlation between corresponding individual samples of the two channel inputs, and for any input signal and noise power levels.

Furthermore, it sometimes happens that the two input channels contain dc components, which can be considered either desirable or otherwise, depending on the application. Here we will consider these dc components as nuisance terms and will subtract them out prior to crosscorrelation. More precisely, since the actual values of the dc components in each channel will generally be unknown, we will estimate them via the sample means (over the available record lengths) and subtract these estimates from the available data. This subtraction feature creates new random variables, all of which are statistically dependent on each other, and thereby significantly complicates the analysis. Nevertheless, this crosscorrelation of the sample ac components of the input channels is encountered in practical situations, and in one recent study [1], it was in fact the generalized likelihood ratio detector under two different realistic scenarios. Accordingly, it merits study and accurate quantitative evaluation of performance capability.

More generally, we consider subtraction of scaled versions of the sample means of each channel prior to multiplication and summation. Then as special cases, we can investigate the crosscorrelator with or without sample mean removal, or any intermediate case of interest.

The major analytical result here is a closed form for the characteristic function of the correlator output, in the most compact form involving only two rooting operations and one exponential. Although this processor could be analyzed by the general method given in [2], in terms of the eigenvalues and eigenvectors of a correlation matrix, it would be less accurate and considerably more time consuming, even with computer aid, especially for a large number of terms summed. The actual numerical procedure adopted here for proceeding from the characteristic function to the exceedance distribution functions (false alarm and detection probabilities) is that given in [3], and utilized to advantage in [2,3,4].

PROBLEM DEFINITION

INPUT STATISTICS

The two channel inputs to the crosscorrelator are synchronously sampled in time, yielding random variables $\{u_n\}_1^N$ and $\{v_n\}_1^N$, where N is the total number of data samples taken in each channel. These random variables are Gaussian with the following statistics:

$$\left.\begin{array}{ll} \text{means} & \overline{u_n} = \mu_u,\ \overline{v_n} = \mu_v, \\ \text{variances} & \overline{(u_n-\overline{u_n})^2} = \sigma_u^2,\ \overline{(v_n-\overline{v_n})^2} = \sigma_v^2, \\ \text{covariances} & \overline{(u_n-\overline{u_n})(v_n-\overline{v_n})} = \rho\sigma_u\sigma_v, \end{array}\right\} \text{all independent of } n. \tag{1}$$

(An overbar denotes a statistical average.) That is, the means and variances in each channel, although different, do not change with time, and the degree of correlation between channels is constant. Also

$$\begin{array}{l} u_m \text{ is statistically independent of } u_n \text{ if } m \neq n, \\ v_m \text{ is statistically independent of } v_n \text{ if } m \neq n, \\ u_m \text{ is statistically independent of } v_n \text{ if } m \neq n. \end{array} \tag{2}$$

However, u_n and v_n are statistically dependent on each other, for all n, to the extent ρ indicated in (1).

A SIGNAL AND NOISE MODEL

To better fix the mathematical definitions above, consider in this subsection the following possible signal and noise model:

$$\left.\begin{array}{l} u_n = \mu_u + u_s(n) + u_d(n) \\ v_n = \mu_v + v_s(n) + v_d(n) \end{array}\right\} \quad \text{for } 1 \leq n \leq N, \tag{3}$$

where random signal components $u_s(n)$, $v_s(n)$ are zero-mean and partially correlated with each other:

$$\left.\begin{aligned} \overline{u_s(n)} = 0,\ \overline{v_s(n)} = 0, \\ \overline{u_s^2(n)} = S_u,\ \overline{v_s^2(n)} = S_v,\ \overline{u_s(n)\ v_s(n)} = \rho_s(S_uS_v)^{1/2} \end{aligned}\right\} \text{for all n.} \quad (4)$$

Thus S_u, S_v are the powers of the random signal components in each channel.

Also, the random noise disturbances $u_d(n)$, $v_d(n)$ in (3) are zero-mean and independent of each other:

$$\left.\begin{aligned} \overline{u_d(n)} = 0,\ \overline{v_d(n)} = 0, \\ \overline{u_d^2(n)} = D_u,\ \overline{v_d^2(n)} = D_v,\ \overline{u_d(n)\ v_d(n)} = 0 \end{aligned}\right\} \text{for all n.} \quad (5)$$

Thus D_u, D_v are the powers of the random noise disturbances in each channel. Finally, except for the statistical dependencies indicated in (4) between $u_s(n)$ and $v_s(n)$, all the 4N random components in (3) are independent of each other.

For this particular signal and noise model in (3)-(5), the master parameters in (1) take the special form

$$\sigma_u^2 = S_u+D_u,\ \sigma_v^2 = S_v+D_v,\ \rho\sigma_u\sigma_v = \rho_s(S_uS_v)^{1/2}\ , \quad (6)$$

from which there follows

$$\rho = \rho_s\left(\frac{R_u}{1+R_u}\,\frac{R_v}{1+R_v}\right)^{1/2}\ , \quad (7)$$

where the signal-to-noise ratios (per sample) of the random components in (3) have been defined as

$$R_u = \frac{S_u}{D_u} = \frac{\overline{u_s^2(n)}}{\overline{u_d^2(n)}}, \quad R_v = \frac{S_v}{D_v} = \frac{\overline{v_s^2(n)}}{\overline{v_d^2(n)}} \quad \text{for all } n. \tag{8}$$

Thus the parameters σ_u, σ_v, ρ in (1) depend only on the statistics of the random components in model (3), and not on the dc components μ_u and μ_v. Observe that even if $\rho_s=1$ and $R_u=\infty$, ρ would still be less than 1; the one noisy channel prevents full correlation between inputs.

CROSSCORRELATOR OUTPUT

We define the sample ac components of each channel of the crosscorrelator as

$$\left.\begin{aligned} \tilde{u}_n &= u_n - \frac{1}{N}\sum_{m=1}^{N} u_m \\ \tilde{v}_n &= v_n - \frac{1}{N}\sum_{m=1}^{N} v_m \end{aligned}\right\} \text{for } 1 \leq n \leq N, \tag{9}$$

where we have subtracted the corresponding sample means from each and every data sample. Thus $\{\tilde{u}_n\}_1^N$ and $\{\tilde{v}_n\}_1^N$ have zero-means and have statistics completely independent of the unknown actual values of input means μ_u, μ_v. However, in trade, we now must deal with a new set of 2N random variables, all of which are statistically dependent on each other; this is the feature which complicates the ensuing analysis. The test statistic (decision variable) of interest is the crosscorrelator output after sample mean removal,

$$q = \sum_{n=1}^{N} \tilde{u}_n \tilde{v}_n = \sum_{n=1}^{N} u_n v_n - \frac{1}{N}\sum_{m=1}^{N} u_m \sum_{n=1}^{N} v_n \,, \tag{10}$$

which is independent of the actual unknown values of input means μ_u and μ_v. If we knew the input means, we could subtract them directly and not have to resort to sample means.

More generally, we consider the modified channel components

$$\left.\begin{aligned} \tilde{u}_n &= u_n - \frac{\alpha}{N}\sum_{m=1}^{N} u_m \\ \tilde{v}_n &= v_n - \frac{\beta}{N}\sum_{m=1}^{N} v_m \end{aligned}\right\} \text{ for } 1 \le n \le N \tag{11}$$

and the crosscorrelator output

$$q = \sum_{n=1}^{N} \tilde{u}_n \tilde{v}_n = \sum_{n=1}^{N} u_n v_n - \frac{\gamma}{N} \sum_{m=1}^{N} u_m \sum_{n=1}^{N} v_n \ , \tag{12}$$

instead of (9) and (10). Scale factors α and/or β in (11) may be unequal to 1; the final parameter γ in (12) is given by

$$\gamma = \alpha + \beta - \alpha\beta = 1 - (\alpha-1)(\beta-1). \tag{13}$$

The case of $\gamma=0$ in (12) obviously corresponds to the case of no sample mean removal. On the other hand, if either* $\alpha=1$ or $\beta=1$, then $\gamma=1$, and we have removal of the sample mean; i.e., (12) reduces to (10). We shall be interested here in the analysis of the general case represented by (12), for arbitrary γ.

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* It is demonstrated in appendix A that if scale factor $\alpha=1$ but $\beta\neq1$, correlator output q is completely independent of μ_u, μ_v, β.

CHARACTERISTIC FUNCTION OF CROSSCORRELATOR OUTPUT

DERIVATION

We express the collection of random variables in (1) and (2) in column matrix form according to

$$U = [u_1\ u_2 \ldots u_N]^T, \quad V = [v_1\ v_2 \ldots v_N]^T , \tag{14}$$

where superscript T denotes transpose. The crosscorrelator output q in (12) can then be written as quadratic form

$$q = U^T Q\ V , \tag{15}$$

where: NxN matrix

$$Q = I - \frac{\gamma}{N} 1\ 1^T , \tag{16}$$

I is the NxN identity matrix, and

$$1 = [1 \ \ 1 \ldots 1]^T \tag{17}$$

is a Nx1 column matrix of ones.

Since U and V are Gaussian, their joint probability density function is, in terms of the parameters in (1),

$$p(U,V) = \left[2\pi\sigma_u\sigma_v\left(1-\rho^2\right)^{1/2}\right]^{-N} \exp\left[-\frac{1}{2(1-\rho^2)} * \right.$$

$$\left. * \left\{\frac{1}{\sigma_u^2}(U - \mu_u 1)^T(U - \mu_u 1) + \frac{1}{\sigma_v^2}(V - \mu_v 1)^T(V - \mu_v 1) - \frac{2\rho}{\sigma_u\sigma_v}(U - \mu_u 1)^T(V - \mu_v 1)\right\}\right] . \tag{18}$$

The characteristic function of correlator output q in (15) is then given by the statistical average

$$f_q(\xi) = \overline{\exp(i\xi q)} = \overline{\exp(i\xi U^T QV)} =$$

$$= \iint dU\ dV\ p(U,V)\ \exp(i\xi U^T QV) =$$

$$= \left[2\pi\sigma_u\sigma_v\left(1-\rho^2\right)^{1/2}\right]^{-N} \iint dU\ dV\ \exp\left[i\xi U^T QV - \frac{1}{2(1-\rho^2)} *\right.$$

$$\left. * \left\{\frac{1}{\sigma_u^2}(U - \mu_u 1)^T (U - \mu_u 1) + \frac{1}{\sigma_v^2}(V - \mu_v 1)^T (V - \mu_v 1) - \frac{2\rho}{\sigma_u\sigma_v}(U - \mu_u 1)^T (V - \mu_v 1)\right\}\right] . \quad (19)$$

At this point, in order to evaluate this 2N-fold integral, we employ the general integral result (B-2) and (B-6) in appendix B, identifying the matrices there as

$$A = \frac{1}{\sigma_u^2(1-\rho^2)}\ I, \quad B = \frac{1}{\sigma_v^2(1-\rho^2)}\ I, \quad C = i\xi Q + \frac{\rho}{\sigma_u\sigma_v(1-\rho^2)}\ I,$$

$$D = \frac{\sigma_v\mu_u - \rho\sigma_u\mu_v}{\sigma_u^2\sigma_v(1-\rho^2)}\ 1, \quad E = \frac{\sigma_u\mu_v - \rho\sigma_v\mu_u}{\sigma_u\sigma_v^2(1-\rho^2)}\ 1. \qquad (20)$$

We also need the following auxiliary results for special matrix forms; namely, for arbitrary scalars c_1, c_2, the matrix determinant

$$\det(c_1 I + c_2\ 1\ 1^T) = c_1^{N-1}(c_1 + Nc_2), \qquad (21)$$

and the matrix inverse

$$(c_1 I + c_2\ 1\ 1^T)^{-1} = \frac{1}{c_1}\ I - \frac{c_2}{c_1(c_1 + Nc_2)}\ 1\ 1^T . \qquad (22)$$

Employment of appendix B and (20)-(22) then yields, after a very considerable amount of effort, a closed form for the characteristic function in (19) (in its most compact form)

$$f_q(\xi) = \frac{\exp\left[i\xi \dfrac{G_1 + i\xi G_2}{1 - i\xi F_1 + \xi^2 F_2}\right]}{\left(1 - i\xi E_1 + \xi^2 E_2\right)^{\frac{N-1}{2}} \left(1 - i\xi F_1 + \xi^2 F_2\right)^{\frac{1}{2}}} , \tag{23}$$

where

$$E_1 = 2\rho\sigma_u\sigma_v, \quad E_2 = \sigma_u^2\sigma_v^2(1-\rho^2) ,$$

$$F_1 = E_1(1-\gamma), \quad F_2 = E_2(1-\gamma)^2 ,$$

$$G_1 = N(1-\gamma)\mu_u\mu_v, \quad G_2 = \tfrac{1}{2}N(1-\gamma)^2(\sigma_u^2\mu_v^2 + \sigma_v^2\mu_u^2 - 2\rho\sigma_u\sigma_v\mu_u\mu_v). \tag{24}$$

The square roots in (23) are principal value, being +1 at $\xi=0$. This characteristic function has four branch points and two essential singularities which overlap two of the branch points; the complexity of this characteristic function of q precludes tractable analytical results for the probability density function or exceedance distribution function of the correlator output, except in very special cases. Nevertheless, since the characteristic function in (23) is easily numerically evaluated with computer aid, it readily lends itself to the procedure presented in [2,3]. A program for the evaluation of the cumulative and exceedance distribution functions corresponding to characteristic function (23)-(24) is given in appendix C for arbitrary values of

- N, number of terms summed
- γ, scale factor in sample mean removal
- μ_u, mean in u-channel
- μ_v, mean in v-channel
- σ_u, standard deviation in u-channel
- σ_v, standard deviation in v-channel
- ρ, correlation coefficient between channels.

A sample plot of the cumulative and exceedance distribution functions for a typical selection of numerical values for the above parameters is also presented in appendix C.

CUMULANTS OF CORRELATOR OUTPUT

By taking the natural logarithm of the characteristic function in (23) and expanding in a power series in ξ, the cumulants of random variable q can be extracted:

$$\chi_q(n) = \tfrac{1}{2}(n-1)!\,[N-1+(1-\gamma)^n]\,(\sigma_u\sigma_v)^n(S^n + D^n) +$$
$$+ \tfrac{1}{2}n!\,N(1-\gamma)^n(\sigma_u\sigma_v)^{n-1}\,\mu_u\mu_v(S^{n-1} + D^{n-1}) +$$
$$+ \tfrac{1}{4}n!\,N(1-\gamma)^n(\sigma_u\sigma_v)^{n-2}(\sigma_u^2\mu_v^2 + \sigma_v^2\mu_u^2)(S^{n-1} - D^{n-1}) \,, \qquad (25)$$

where here

$$S = \rho + 1, \quad D = \rho - 1 \,. \qquad (26)$$

In particular, the mean and variance of q are available by using n=1 and 2 respectively in (25):

$$\mu_q = (N-\gamma)\rho\sigma_u\sigma_v + N(1-\gamma)\mu_u\mu_v,$$
$$\sigma_q^2 = (N-2\gamma+\gamma^2)(1+\rho^2)\sigma_u^2\sigma_v^2 + N(1-\gamma)^2(\sigma_u^2\mu_v^2 + \sigma_v^2\mu_u^2 + 2\rho\sigma_u\sigma_v\mu_u\mu_v). \qquad (27)$$

SPECIAL CASE OF γ=1, SAMPLE MEAN REMOVAL

For γ=1, the general characteristic function in (23) reduces to

$$f_q(\xi;\ \gamma=1) = \left(1-i\xi E_1 + \xi^2 E_2\right)^{-\frac{N-1}{2}} \,, \qquad (28)$$

where E_1 and E_2 are still given by (24), and are independent of means μ_u and μ_v, as shown earlier. The cumulants in (25) reduce to

$$\chi_q(n) = \frac{1}{2}(n-1)!(N-1)(\sigma_u\sigma_v)^n[(\rho+1)^n + (\rho-1)^n] , \tag{29}$$

and in particular, the mean and variance of q are

$$\mu_q = (N-1)\rho\sigma_u\sigma_v,$$

$$\sigma_q^2 = (N-1)(1+\rho^2)\sigma_u^2\sigma_v^2 . \tag{30}$$

SPECIAL CASE OF γ=0, SAMPLE MEAN NOT REMOVED

For γ=0, the characteristic function in (23) reduces to

$$f_q(\xi;\gamma=0) = \left(1 - i\xi E_1 + \xi^2 E_2\right)^{-N/2} \exp\left[i\xi \frac{G_1^{(o)} + i\xi G_2^{(o)}}{1-i\xi E_1 + \xi^2 E_2}\right], \tag{31}$$

where

$$E_1 = 2\rho\sigma_u\sigma_v, \quad E_2 = \sigma_u^2\sigma_v^2(1-\rho^2),$$

$$G_1^{(o)} = N\mu_u\mu_v, \quad G_2^{(o)} = \frac{1}{2}N(\sigma_u^2\mu_v^2 + \sigma_v^2\mu_u^2 - 2\rho\sigma_u\sigma_v\mu_u\mu_v). \tag{32}$$

The cumulants are obtained by setting γ=0 in (25), and in particular, the mean and variance of correlator output q are

$$\mu_q = N(\rho\sigma_u\sigma_v + \mu_u\mu_v),$$

$$\sigma_q^2 = N\left[(1+\rho^2)\sigma_u^2\sigma_v^2 + \sigma_u^2\mu_v^2 + \sigma_v^2\mu_u^2 + 2\rho\sigma_u\sigma_v\mu_u\mu_v\right]. \tag{33}$$

INTERRELATIONSHIP OF TWO SPECIAL CASES

Let the general characteristic function in (23) be denoted by $f_q(\xi;\ N,\ \gamma,\ \mu_u,\ \mu_v)$. We have already seen the expression for $f_q(\xi;\ N,\ 1,\ \mu_u,\ \mu_v)$ in (28). At the same time, from (23) and (24), there follows

$$f_q(\xi;\ N\text{-}1,\ 0,\ 0,\ 0,) = \left(1 - i\xi E_1 + \xi^2 E_2\right)^{-\frac{N-1}{2}}, \tag{34}$$

which is identical to (28). That is,

$$f_q(\xi;\ N,\ 1,\ \mu_u,\ \mu_v) = f_q(\xi;\ N\text{-}1,\ 0,\ 0,\ 0). \tag{35}$$

Thus the characteristic functions of the two following random variables are identical:

(1) Sum of N terms with sample mean removal, and the true means arbitrary,

(2) Sum of N-1 terms without sample mean removal, but the true means zero. (36)

The removal of the sample means has eliminated the dependence of the correlator output on the unknown means but has reduced the number of degrees of freedom by 1.

SPECIALIZATION TO THE SIGNAL AND NOISE MODEL

For general scaling factor γ and arbitrary input means μ_u, μ_v, and for the model introduced earlier in (3)-(6), the general characteristic function of the correlator output is still given by (23), but with the parameters in (24) now specialized to the form

$$E_1 = 2\rho_s(S_uS_v)^{1/2}, \quad E_2 = D_uD_v+D_uS_v+D_vS_u+S_uS_v(1-\rho_s^2),$$

$$F_1 = E_1(1-\gamma), \qquad F_2 = E_2(1-\gamma)^2,$$

$$G_1 = N(1-\gamma)\mu_u\mu_v, \; G_2 = \frac{1}{2}N(1-\gamma)^2[(S_u+D_u)\mu_v^2+(S_v+D_v)\mu_u^2-2\rho_s(S_uS_v)^{1/2}\mu_u\mu_v] \; . \qquad (37)$$

The general n-th cumulant is still given by (25); however, the use of (6) allows for determination in terms of the fundamental quantities of the signal and noise model, namely S_u, S_v, D_u, D_v, ρ_s defined in (4)-(5). In particular, the mean and variance of correlator output q are

$$\mu_q = (N-\gamma)\rho_s(S_uS_v)^{1/2} + N(1-\gamma)\mu_u\mu_v,$$

$$\sigma_q^2 = (N-2\gamma+\gamma^2)[D_uD_v+D_uS_v+D_vS_u+(1+\rho_s^2)S_uS_v]+$$

$$+ N(1-\gamma)^2[(S_u+D_u)\mu_v^2+(S_v+D_v)\mu_u^2+2\rho_s(S_uS_v)^{1/2}\mu_u\mu_v]. \qquad (38)$$

ANALYTIC RESULTS FOR γ=1, SAMPLE MEAN REMOVAL

In this section and the next, we will confine attention solely to the case of scale factor $\gamma=1$. The characteristic function of the crosscorrelator output q follows from (28) and (24) as

$$f_q(\xi) = [1 - i\xi E_1 + \xi^2 E_2]^{-\frac{N-1}{2}} = [1 - i\xi 2\rho\sigma_u\sigma_v + \xi^2\sigma_u^2\sigma_v^2(1-\rho^2)]^{-\frac{N-1}{2}} =$$

$$= \left\{[1+i\xi\sigma_u\sigma_v(1-\rho)]\,[1-i\xi\sigma_u\sigma_v(1+\rho)]\right\}^{-\frac{N-1}{2}} \quad \text{for } \gamma=1 , \qquad (39)$$

where we must have $N\geq 2$. We observe, for later numerical use in appendix D, that since $|1 \pm i\xi b| = (1 + \xi^2 b^2)^{1/2}$ is monotonically increasing for $\xi\geq 0$, then $|f_q(\xi)|$ is monotonically decreasing for all $\xi\geq 0$ and any N, σ_u, σ_v, ρ.

GENERAL PROBABILITY RESULTS

The cumulants of q have already been listed in (29) and (30). The probability density function corresponding to characteristic function (39) is given by [5, 6.699 12]

$$p_q(u) = \left[\Gamma\left(\frac{N-1}{2}\right) \pi^{1/2}(1-\rho^2)^{1/2}\sigma_u\sigma_v\right]^{-1} \left(\frac{|u|}{2\sigma_u\sigma_v}\right)^{\frac{N}{2}-1} *$$

$$* K_{\frac{N}{2}-1}\left(\frac{|u|}{\sigma_u\sigma_v(1-\rho^2)}\right)\exp\left(\frac{\rho u}{\sigma_u\sigma_v(1-\rho^2)}\right) \quad \text{for all } u, \quad \gamma=1 , \qquad (40)$$

where $K_\nu(z)$ is a modified Bessel function of the second kind [6, section 9.6]. If the number of terms added, N, to yield correlator output q, is odd, simple relations for the probability density function in (40) can be obtained [6, 10.2.15 and 10.1.9, last equation]; letting $n = \frac{N-3}{2}$ for N odd, we find the exact result

$$p_q(u) = \frac{(1-\rho^2)^n}{2\sigma_u\sigma_v\, 4^n\, n!} \exp\left(\frac{\rho u - |u|}{\sigma_u\sigma_v(1-\rho^2)}\right) \sum_{m=0}^{n} \frac{(2n-m)!}{(n-m)!\ m!} \left(\frac{2\,|u|}{\sigma_u\sigma_v(1-\rho^2)}\right)^m$$

$$\text{for all } u;\ n = \frac{N-3}{2},\ N = 3, 5, 7, \ldots \,. \tag{41}$$

For example, for N=3, we have n=0, yielding

$$p_q(u) = \frac{1}{2\sigma_u\sigma_v} \exp\left(\frac{\rho u - |u|}{\sigma_u\sigma_v(1-\rho^2)}\right) \quad \text{for all } u. \tag{42}$$

The corresponding cumulative distribution function for N=3 is

$$P_q(u) = \int_{-\infty}^{u} dt\ p_q(t) = \frac{1-\rho}{2} \exp\left(\frac{u}{\sigma_u\sigma_v(1-\rho)}\right) \quad \text{for } u \le 0, \tag{43A}$$

while the exceedance distribution function is

$$1 - P_q(u) = \int_{u}^{+\infty} dt\ p_q(t) = \frac{1+\rho}{2} \exp\left(\frac{-u}{\sigma_u\sigma_v(1+\rho)}\right) \quad \text{for } u \ge 0. \tag{43B}$$

This dichotomy, of presenting the cumulative distribution function for negative arguments, and the exceedance distribution function for positive arguments, turns out to be notationally convenient and physically meaningful and will be adopted throughout this report.

Although closed form expressions for the exceedance distribution function corresponding to probability density function (40) are not available for general N, the use of [6, 9.7.2] on (40) leads to the dominant term in the asymptotic expansion of the exceedance distribution function:

$$1 - P_q(u) \sim \frac{1+\rho}{2\Gamma\left(\frac{N-1}{2}\right)} \left(\frac{u}{2\sigma_u\sigma_v}\right)^{\frac{N-3}{2}} \exp\left(\frac{-u}{\sigma_u\sigma_v(1+\rho)}\right) \quad \text{as } u \to +\infty\,. \tag{44}$$

For N=3, this is precise; see (43B).

POSSIBLE NORMALIZATIONS OF q

If we define a normalized random variable

$$x = \frac{q}{E_2^{1/2}} = \frac{q}{\sigma_u \sigma_v (1-\rho^2)^{1/2}} , \tag{45}$$

then the characteristic function of x is given by (39) as

$$f_x(\xi) = f_q(\xi/E_2^{1/2}) = \left[1 - i\xi E_1/E_2^{1/2} + \xi^2\right]^{-\frac{N-1}{2}} =$$

$$= \left[1 - i\xi 2\rho(1-\rho^2)^{-1/2} + \xi^2\right]^{-\frac{N-1}{2}} , \tag{46}$$

which has only two fundamental parameters, namely, N and ρ.

A second possibility is the random variable defined by

$$y = \frac{q}{\sigma_u \sigma_v} , \tag{47}$$

for which characteristic function

$$f_y(\xi) = f_q\left(\frac{\xi}{\sigma_u \sigma_v}\right) = \left[1 - i\xi 2\rho + \xi^2(1-\rho^2)\right]^{-\frac{N-1}{2}} \tag{48}$$

also depends only on N and ρ. However, neither of the normalizations, (45) and (47), are of interest to us here; an alternative normalization and reasons for its selection are given below.

SPECIALIZATION TO THE SIGNAL AND NOISE MODEL

For the model presented earlier in (3)-(8), the original E_1, E_2 parameters in (24) take the form already given in the upper line of (37). Let a normalized random variable, relative to the additive random noise disturbances, be defined according to

$$h = \frac{q}{(D_u D_v)^{1/2}} ; \tag{49}$$

see (5). This normalization for the particular signal model (3) is different from both x and y in the general case above. The reason we employ h is that the normalization depends only on the power of the additive noise disturbances, and not on the signal strengths or correlation coefficients; this is consistent with a system which monitors the noise-only background and sets a threshold for a desired false alarm probability.

The characteristic function of the normalized random variable h in (49) is given by

$$f_h(\xi) = f_q\left(\xi/(D_u D_v)^{1/2}\right) = \left[1 - i\xi\frac{E_1}{(D_u D_v)^{1/2}} + \xi^2 \frac{E_2}{D_u D_v}\right]^{-\frac{N-1}{2}} =$$

$$= \left[1 - i\xi 2\alpha + \xi^2(\beta^2 - \alpha^2)\right]^{-\frac{N-1}{2}} , \tag{50}$$

where we define auxiliary parameters here as

$$\alpha = \rho_s (R_u R_v)^{1/2} , \quad \beta = [(1+R_u)(1+R_v)]^{1/2} . \tag{51}$$

Here we used (39), (37), and (8). This characteristic function in (50) depends on the four fundamental parameters N, ρ_s, R_u, R_v, where the latter two quantities are the signal-to-noise ratios per sample of the random components of model (3); see (8).

Reference to (40) reveals that the probability density function of h corresponding to characteristic function (50) is given by

$$p_h(u) = \left[\Gamma\left(\frac{N-1}{2}\right) \pi^{1/2} (\beta^2-\alpha^2)^{1/2}\right]^{-1} \left(\frac{|u|}{2\beta}\right)^{\frac{N}{2}-1} *$$

$$* \, K_{\frac{N}{2}-1}\left(\frac{\beta|u|}{\beta^2-\alpha^2}\right) \exp\left(\frac{\alpha u}{\beta^2-\alpha^2}\right) \quad \text{for all } u. \tag{52}$$

For N odd, alternative forms are available from (41), if desired. The asymptotic behavior of the exceedance distribution function of h follows in a manner similar to that used for (44):

$$1 - P_h(u) \sim \frac{1+\alpha/\beta}{2\Gamma\left(\frac{N-1}{2}\right)} \left(\frac{u}{2\beta}\right)^{\frac{N-3}{2}} \exp\left(\frac{-u}{\beta+\alpha}\right) \quad \text{as } u \to +\infty . \tag{53}$$

The cumulants of h follow from (29), (26), and (6)-(8):

$$\chi_h(n) = \tfrac{1}{2}(n-1)!(N-1)[(\alpha+\beta)^n + (\alpha-\beta)^n] , \tag{54}$$

and in particular, the mean and variance of h are

$$\mu_h = (N-1)\alpha = (N-1)\rho_s(R_u R_v)^{1/2} ,$$

$$\sigma_h^2 = (N-1)(\alpha^2+\beta^2) = (N-1)[1 + R_u + R_v + R_u R_v(1+\rho_s^2)] . \tag{55}$$

The two parameters, α and β, are given here by (51), in terms of the fundamental quantities R_u, R_v, ρ_s of the signal and noise model.

REDUCTION TO IDENTICAL SIGNAL COMPONENTS

At this point, we will further specialize the results for the signal and noise model in the above subsection. We presume that

$$R_u = R_v = R \quad \text{and} \quad \rho_s = 1 \ , \tag{56A}$$

giving, from (51),

$$\alpha = R, \quad \beta = 1+R; \tag{56B}$$

that is, the signal-to-noise ratios in the two channels are equal, and the two channel signals are fully correlated. This corresponds physically to a case where the random signal components in (3) are identical, $u_s(n) = v_s(n)$, and the independent random noise disturbances have the same power level. This situation will hold for the rest of this section and all of the next section where the graphical results are presented.

Equations (50) and (56B) then yield the characteristic function for normalized random variable h in (49) as

$$f_h(\xi) = [1 - i\xi 2R + \xi^2(1+2R)]^{-\frac{N-1}{2}} =$$

$$= [(1 + i\xi)(1 - i\xi(1+2R))]^{-\frac{N-1}{2}} . \tag{57}$$

The cumulants in (54) reduce to

$$\chi_h(n) = \frac{1}{2}(n-1)!(N-1)[(1+2R)^n+(-1)^n] \ , \tag{58}$$

and in particular, the mean and variance become

$$\mu_h = (N-1)R, \quad \sigma_h^2 = (N-1)(1+2R+2R^2) \ . \tag{59}$$

All the above statistical descriptions depend only on the two parameters R, the per-sample signal-to-noise ratio, and N, the number of terms added to yield the correlator output.

The probability density function for h follows from (52) as

$$p_h(u) = \left[\Gamma\left(\frac{N-1}{2}\right) \pi^{1/2} (1+2R)^{1/2}\right]^{-1} \left(\frac{|u|}{2(1+R)}\right)^{\frac{N}{2}-1} *$$

$$* \; K_{\frac{N}{2}-1}\left(\frac{1+R}{1+2R}|u|\right) \exp\left(\frac{Ru}{1+2R}\right) \quad \text{for all } u \, , \tag{60}$$

and the asymptotic exceedance distribution function from (53):

$$1 - P_h(u) \sim \frac{1+2R}{2(1+R)\Gamma\left(\frac{N-1}{2}\right)} \left(\frac{u}{2(1+R)}\right)^{\frac{N-3}{2}} \exp\left(\frac{-u}{1+2R}\right) \quad \text{as } u \to +\infty \, . \tag{61}$$

An important word of caution must be mentioned at this point: when N is large, (61) is inadequate for evaluating small false alarm and detection probabilities, since the succeeding terms in the asymptotic expansion contribute significantly. For example, when $R=0$, the maximum value of the dominant term (61) occurs when $u = (N-3)/2$ which, for $N=128$, yields false alarm probability 3.86E-21, a value far below those of interest. Thus (61) has limited applicability, being best for small N; in fact, the first correction term to (61) yields the multiplicative factor

$$1 + \frac{1+2R}{1+R} \frac{(N-3)(N+3+4R)}{8u} \, . \tag{62}$$

It indicates that, for large N, u must be of the order of N^2 in order for the dominant term (61) to be fairly accurate.

Although for N odd, an alternative closed form to the probability density function (60) of h is available from (41), the exceedance distribution function will generate a double sum and be rather cumbersome for large N. On the other hand, the characteristic function in (57) decays rapidly with ξ when N is large and yields very nicely to the numerical approach given in [2,3]. The only difficult cases are in fact those for small N; accordingly, some analytic results for N = 2, 3, 4, 5 will now be presented, based on characteristic function (57).

GENERAL DISTRIBUTION INTEGRALS

Suppose a random variable y has characteristic function $f_y(\xi)$. The cumulative distribution function of y can be written as a contour integral [3, (5) & (6)]

$$P_y(u) = -\frac{1}{i2\pi} \int_{C_+} d\xi \, \frac{f_y(\xi)}{\xi} \exp(-iu\xi) \quad \text{for all } u \; , \tag{63}$$

where C_+ is a contour along the real axis of the complex ξ-plane, with an upward indentation at the origin $\xi=0$, to avoid the pole of the integrand there.

Similarly, the exceedance distribution function of random variable y can be expressed as

$$1 - P_y(u) = \frac{1}{i2\pi} \int_{C_-} d\xi \, \frac{f_y(\xi)}{\xi} \exp(-iu\xi) \quad \text{for all } u \; , \tag{64}$$

where C_- is a contour along the real ξ axis, with a downward indentation at $\xi=0$.

For $u<0$, both contours can be closed in the upper-half ξ-plane, since the exp term furnishes rapid decay there. Similarly, for $u>0$, both contours can be closed in the lower-half ξ-plane, to realize exponential decay on the circular arcs tending to infinity.

DISTRIBUTIONS FOR N=2

From (57), the characteristic function of normalized correlator output h is

$$f_h(\xi) = [(1+i\xi)(1-i\xi(1+2R))]^{-1/2} \; , \tag{65}$$

and the probability density function follows from (60) as

$$p_h(u) = \frac{1}{\pi(1+2R)^{1/2}} \; K_o\left(\frac{1+R}{1+2R}|u|\right) \exp\left(\frac{Ru}{1+2R}\right) \quad \text{for all } u \; . \tag{66}$$

There is no closed form for the indefinite integral of a K_o function; see [6, 11.1.8 and 11.1.9]. Instead, we use (65) in (63) and move the contour upwards until it wraps around the branch point at $\xi=i$ and extends vertically from there; this is in fact the steepest descent direction for the exponential. The contributions of the small and large circular arcs tend to zero as the radii tend to zero and infinity, respectively. Under a change of variable, there follows the cumulative distribution function in the form

$$P_h(u) = \frac{2}{\pi}\int_0^{+\infty} \frac{dt\; \exp[u(1+t^2)]}{(1+t^2)[1+(1+2R)(1+t^2)]^{1/2}} \quad \text{for } u \le 0 \; . \tag{67}$$

This is a useful exact result for several reasons: the integrand decays rapidly, has no cusps, and involves only elementary functions which are easily computed; also the integral is a sum of positive quantities and retains significance even for large $|u|$.

In a similar fashion, if characteristic function (65) is substituted in (64) and the contour moved down and wrapped around the branch point at $\xi = -i/(1+2R)$ and along the vertical steepest descent direction for the exponential, the exceedance distribution function becomes, upon a change of variable,

$$1 - P_h(u) = \frac{2}{\pi}(1+2R)\; \exp\left(\frac{-u}{1+2R}\right) \int_0^{+\infty} \frac{dt\; \exp(-ut^2)}{[1+(1+2R)t^2][1+(1+2R)(1+t^2)]^{1/2}} \quad \text{for } u \ge 0. \tag{68}$$

This is useful for the same reasons given above.

There is one closed form result possible; namely, for u=0, direct integration of probability density function (66) yields [5, 6.611 9]

$$P_h(0) = \frac{1}{\pi} \text{arc cos}\left(\frac{R}{1+R}\right) \; , \; 1-P_h(0) = \frac{1}{\pi} \text{arc cos}\left(\frac{-R}{1+R}\right) \; . \tag{69}$$

DISTRIBUTIONS FOR N=3

Use of [6, 10.2.17] on (60) immediately yields probability density function

$$p_h(u) = \begin{cases} \dfrac{1}{2(1+R)} \exp(u) & \text{for } u \le 0 \\ \\ \dfrac{1}{2(1+R)} \exp\left(\dfrac{-u}{1+2R}\right) & \text{for } u \ge 0 \end{cases} . \tag{70}$$

The cumulative and exceedance distribution functions easily follow as

$$\begin{aligned} P_h(u) &= \frac{1}{2(1+R)} \exp(u) & \text{for } u \le 0 , \\ 1 - P_h(u) &= \frac{1+2R}{2(1+R)} \exp\left(\frac{-u}{1+2R}\right) & \text{for } u \ge 0 . \end{aligned} \tag{71}$$

This latter result corroborates (61) and (62).

DISTRIBUTIONS FOR N=4

The only closed form result possible is obtained by direct integration of probability density function (60) to get origin value

$$P_h(0) = \frac{1}{\pi} \left[\text{arc cos}\left(\frac{R}{1+R}\right) - \frac{R(1+2R)^{1/2}}{(1+R)^2}\right] . \tag{72}$$

This follows by use of the integral

$$\int_0^{+\infty} dx\; e^{-\alpha x}\, x\, K_1(\beta x) = \frac{\beta \text{ arc cos}(\alpha/\beta)}{(\beta^2-\alpha^2)^{3/2}} - \frac{\alpha}{\beta(\beta^2-\alpha^2)} \quad \text{for } \beta > -\alpha , \tag{73}$$

which follows from [5, 6.611 9] by applying $\partial/\partial\beta$ to both sides.

DISTRIBUTIONS FOR N=5

Use of [6, 10.2.17] on (60) immediately yields probability density function

$$p_h(u) = \begin{cases} \dfrac{1+2R-(1+R)u}{4(1+R)^3} \exp(u) & \text{for } u \le 0 \\[2ex] \dfrac{1+2R+(1+R)u}{4(1+R)^3} \exp\left(\dfrac{-u}{1+2R}\right) & \text{for } u \ge 0 \end{cases} . \tag{74}$$

The cumulative and exceedance distribution functions follow as

$$P_h(u) = \frac{2+3R-(1+R)u}{4(1+R)^3} \exp(u) \quad \text{for } u \le 0,$$

$$1 - P_h(u) = \frac{1+2R}{4(1+R)^3} [(1+2R)(2+R)+(1+R)u] \exp\left(\frac{-u}{1+2R}\right) \text{ for } u \ge 0 . \tag{75}$$

This latter result corroborates (61) and (62). Also, this example was used as a check on the numerical procedure [3] applied directly to the characteristic function, which is used in the following section; the agreement was ten decimals for numerous values of R and u.

GRAPHICAL RESULTS FOR γ=1, SAMPLE MEAN REMOVAL

SUMMARY OF PARTICULAR CASE CONSIDERED

We first summarize here the particular case that will be considered quantitatively in this section. The input samples are

$$\left.\begin{aligned} u_n &= \mu_u + u_s(n) + u_d(n) \\ v_n &= \mu_v + v_s(n) + v_d(n) \end{aligned}\right\} \quad \text{for } 1 \le n \le N, \tag{76}$$

where these Gaussian random variables have statistics

$$\overline{u_s(n)} = \overline{v_s(n)} = \overline{u_d(n)} = \overline{v_d(n)} = 0 \ ,$$

$$\overline{u_s^2(n)} = S_u, \ \overline{v_s^2(n)} = S_v, \ \overline{u_s(n)v_s(n)} = (S_u S_v)^{1/2} \ ,$$

$$\overline{u_d^2(n)} = D_u, \ \overline{v_d^2(n)} = D_v, \ \overline{u_d(n)v_d(n)} = 0 \ . \tag{77}$$

We presume that the simultaneous signal components $u_s(n)$, $v_s(n)$ in the two channels are fully correlated, that all other random variables are independent, and that the two channel input signal-to-noise ratios

$$\frac{S_u}{D_u} = \frac{S_v}{D_v} = R \tag{78}$$

have a common value R. More general situations have been considered in earlier sections; however, only this special case will be numerically evaluated here.

The normalized crosscorrelator output, with sample mean removal ($\gamma=1$), is

$$h = \frac{1}{(D_u D_v)^{1/2}} \sum_{n=1}^{N} \tilde{u}_n \tilde{v}_n , \tag{79}$$

where the sample ac components

$$\tilde{u}_n = u_n - \frac{1}{N} \sum_{m=1}^{N} u_m , \qquad \tilde{v}_n = v_n - \frac{1}{N} \sum_{m=1}^{N} v_m . \tag{80}$$

The characteristic function of h is given by (57) as

$$f_h(\xi) = [(1+i\xi)(1-i\xi(1+2R))]^{-\frac{N-1}{2}} \tag{81}$$

and depends only on signal-to-noise ratio R and number of terms N. We must have $N \geq 2$.

If R=0 and we evaluate the exceedance distribution function corresponding to (81), we then have the false alarm probability. But when R>0, the exceedance distribution function corresponding to (81) is the detection probability. In the following, we plot the detection probability vs. the false alarm probability, with signal-to-noise ratio R as a parameter; different values of N are handled in separate plots.

OPERATING CHARACTERISTICS FOR $\gamma=1$

A sample program for evaluation of the cumulative and exceedance distribution functions corresponding to characteristic function (81), and thereby the detection probability vs. false alarm probability operating characteristics of the crosscorrelator with sample mean removal, is given in appendix D. It is heavily based on the technique developed and explained in [3].

In figures* 1-14 are presented the operating characteristics for the crosscorrelator with sample mean removal, for values of

$$N = 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 64, 96, 128, 256, \tag{82}$$

respectively. The case of N=2 was accomplished by use of (67)-(69); results for N=3 relied on (71); and the remainder for $N \geq 4$ employed a numerical procedure [3] proceeding directly from characteristic function (81) to the exceedance distribution function. False alarm probabilities P_F in the range 1E-10 to .5 and detection probabilities P_D covering 1E-10 to .999 are presented. The abscissa and ordinate on these plots are according to a normal probability transformation, as explained below. Values of signal-to-noise ratio R are taken as $R=2^n$, where n assumes values appropriate for each plot in order to cover the full range of probabilities of interest.

GAUSSIAN APPROXIMATION

Suppose the decision variable of a processor is Gaussian with mean and standard deviation m_o, σ_o respectively when the input signal is absent, and m_1, σ_1 when signal is present. Then for threshold Λ, the false alarm probability and detection probability are

$$P_F = \int_{\Lambda}^{+\infty} du \, \frac{1}{\sigma_o} \, \phi\left(\frac{u-m_o}{\sigma_o}\right) = \Phi\left(\frac{m_o-\Lambda}{\sigma_o}\right),$$

$$P_D = \int_{\Lambda}^{+\infty} du \, \frac{1}{\sigma_1} \, \phi\left(\frac{u-m_1}{\sigma_1}\right) = \Phi\left(\frac{m_1-\Lambda}{\sigma_1}\right), \tag{83}$$

respectively, where ϕ and Φ are the normalized Gaussian probability density function and cumulative distribution function:

$$\phi(u) = (2\pi)^{-1/2} \exp(-u^2/2), \quad \Phi(u) = \int_{-\infty}^{u} dt \, \phi(t) \, . \tag{84}$$

If we let Φ_I be the inverse function to Φ, and define

$$x = \Phi_I(P_F), \quad y = \Phi_I(P_D) \, , \tag{85}$$

*See page 51 et seq.

then threshold Λ can be eliminated from (83) to yield

$$y = \frac{m_1 - m_0 + \sigma_0 x}{\sigma_1} \; . \tag{86}$$

Equation (85) corresponds to the transformation to normal probability coordinates; thus a plot of P_D vs P_F on normal probability paper is the straight line (86) when the decision variable is Gaussian under both hypotheses of signal absent as well as present.

Reference to (59) reveals that, for our application,

$$m_0 = 0, \quad m_1 = (N-1)R, \quad \sigma_0^2 = N-1, \quad \sigma_1^2 = (N-1)(1+2R+2R^2) \; , \tag{87}$$

since setting signal-to-noise ratio R=0 corresponds to hypothesis 0, signal absent. Substitution in (86) yields

$$y = \frac{(N-1)^{1/2}\, R + x}{(1+2R+2R^2)^{1/2}} \; ; \tag{88}$$

that is, if normalized crosscorrelator output h were Gaussian, the operating characteristics would be straight lines dictated by (88). These straight lines are superposed as dashed lines in figures 12-14 for N=96, 128, 256 respectively. Despite the large value of N=96 in figure 12, the Gaussian approximation is not that accurate, especially for small false alarm probabilities and large detection probabilities. The exact curve (solid) and the Gaussian approximation (dashed) cross each other, and are labelled at the crossing with the corresponding value of n in signal-to-noise ratio $R=2^n$. For N=256 in figure 14, agreement is better and the Gaussian approximation is probably adequate for larger N. If not, an additional term or two in an Edgeworth expansion could be investigated with the aid of the cumulants given in (58).

An obvious shortcoming of the Gaussian approximation (88) may be seen immediately:

$$\lim_{R \to \infty} y = \left(\frac{N-1}{2}\right)^{1/2} < 1 \quad \text{for any } x \, . \tag{89}$$

Reference to (85) then yields the interpretation

$$\lim_{R \to \infty} P_D = \Phi\left(\left(\frac{N-1}{2}\right)^{1/2}\right) < 1 \quad \text{for any } P_F \, . \tag{90}$$

That is, as input signal-to-noise ratio R tends to infinity, the approximate detection probability saturates at a value less than 1, regardless of the false alarm probability. Thus the Gaussian approximation must certainly deteriorate for large R; the exact discrepancy for probabilities of practical interest is displayed in figures 12-14.

ANALYTIC RESULTS FOR $\gamma=0$, SAMPLE MEAN NOT REMOVED

In this section and the next, attention will be confined solely to the case of scale factor $\gamma=0$. The characteristic function of the crosscorrelator output q is then given in (31) and (32), and the mean and variance of q are listed in (33). Due to the complexity of the exponential term in characteristic function (31), there are no general probability density function or cumulative distribution function results for arbitrary N, like those given earlier in (40), (41), and (44) for $\gamma=1$. Here, we can have $N \geq 1$.

SPECIALIZATION TO THE SIGNAL AND NOISE MODEL

For the model presented earlier in (3)-(8), the original parameters in (24) take the form already given in (37), but now with $\gamma=0$. Specifically, characteristic function (31) is

$$f_q(\xi) = \left(1 - i\xi E_1 + \xi^2 E_2\right)^{-N/2} \exp\left[i\xi \frac{G_1^{(o)} + i\xi G_2^{(o)}}{1-i\xi E_1 + \xi^2 E_2}\right], \tag{91}$$

where

$$E_1 = 2\rho_s(S_u S_v)^{1/2}, \quad E_2 = D_u D_v + D_u S_v + D_v S_u + S_u S_v(1-\rho_s^2) ,$$

$$G_1^{(o)} = N\mu_u\mu_v, \quad G_2^{(o)} = \tfrac{1}{2}N[(S_u+D_u)\mu_v^2+(S_v+D_v)\mu_u^2-2\rho_s(S_u S_v)^{1/2}\mu_u\mu_v]. \tag{92}$$

The mean and variance of crosscorrelator output q follow from (38) according to

$$\mu_q = N[\rho_s(S_u S_v)^{1/2} + \mu_u\mu_v] ,$$

$$\sigma_q^2 = N[D_u D_v+D_u S_v+D_v S_u+(1+\rho_s^2)S_u S_v + $$

$$+(S_u+D_u)\mu_v^2+(S_v+D_v)\mu_u^2+2\rho_s(S_u S_v)^{1/2}\mu_u\mu_v] . \tag{93}$$

NORMALIZED CROSSCORRELATOR OUTPUT

As in (49), and for the same reasons, we define a normalized crosscorrelator output, relative to the additive random noise disturbances $u_d(n)$ and $v_d(n)$ in (3) and (5), according to

$$h = \frac{q}{(D_u D_v)^{1/2}} . \tag{94}$$

The characteristic function of h is available from (91) and (92):

$$f_h(\xi) = f_q\left(\xi/(D_u D_v)^{1/2}\right) =$$

$$= [1-i\xi 2\alpha+\xi^2(\beta^2-\alpha^2)]^{-N/2} \exp\left[i\xi \frac{a+i\xi b}{1-i\xi 2\alpha+\xi^2(\beta^2-\alpha^2)}\right], \tag{95}$$

where

$$\alpha = \rho_s(R_u R_v)^{1/2} , \quad \beta = [(1+R_u)(1+R_v)]^{1/2} ,$$

$$a = Nr_u r_v, \; b = \tfrac{1}{2}N[(1+R_u)r_v^2 + (1+R_v)r_u^2 - 2\rho_s(R_u R_v)^{1/2} r_u r_v] , \tag{96}$$

and where we have defined

$$R_u = \frac{S_u}{D_u} , \; R_v = \frac{S_v}{D_v} , \; r_u = \frac{\mu_u}{D_u^{1/2}} , \; r_v = \frac{\mu_v}{D_v^{1/2}} . \tag{97}$$

The characteristic function in (95) depends on six fundamental parameters, namely N, ρ_s, R_u, R_v, r_u, r_v. The mean and variance of h follow from (94), (93), and (97):

$$\mu_h = N[\rho_s(R_uR_v)^{1/2} + r_ur_v] \; ,$$

$$\sigma_h^2 = N[(1+R_u)(1+R_v)+\rho_s^2R_uR_v+(1+R_u)r_v^2+(1+R_v)r_u^2+2\rho_s(R_uR_v)^{1/2}\; r_ur_v] \; . \tag{98}$$

r_u and r_v are referred to as normalized means.

REDUCTION TO IDENTICAL SIGNAL COMPONENTS

In order to prepare for numerical evaluation of the operating characteristics of the crosscorrelator with γ=0, we further specialize the signal and noise model to the case where

$$R_u = R_v = R, \quad \rho_s = 1, \quad r_u = r_v = r \; ; \tag{99}$$

see (56) et seq. This leads to

$$\alpha = R, \quad \beta = 1+R, \quad a = b = Nr^2 \; ,$$

via (96). The characteristic function in (95) then reduces to

$$f_h(\xi) = [(1+i\xi)(1-i\xi(1+2R))]^{-N/2} \exp\left[\frac{i\xi Nr^2}{1-i\xi(1+2R)}\right] , \tag{100}$$

and the mean and variance in (98) become

$$\mu_h = N(R+r^2), \quad \sigma_h^2 = N[1+2R+2R^2+2(1+2R)r^2] \; . \tag{101}$$

The characteristic function in (100) has a branch point at $\xi = i$, and another branch point at $\xi = -i/(1+2R)$ which overlaps an essential singularity; this complicates some of the analytical development to follow.

There are three fundamental parameters in (100), namely N, R, r. Since normalized mean r appears only through its square, we can presume $r \geq 0$ without loss of generality. Furthermore, if r=0, characteristic function (100) reduces to (57) if N-1 there is replaced by N. Thus the curves for r=0 here can be obtained from the earlier curves for $\gamma=1$ in figures 1-14 by looking at a value for N there which is one greater; accordingly we can confine attention to r>0 in this and the next section.

When the random signal components $u_s(n)$, $v_s(n)$ in model (3) are absent, then R=0, and the exceedance distribution function corresponding to characteristic function (100) becomes the false alarm probability. However, (100) still depends on r, meaning that the false alarm probability must, likewise. Thus, a non-zero mean in (3), i.e. r>0, is not considered a signal attribute here, but rather is a nuisance quantity; it may, in fact, degrade the operating characteristics if not removed.

For notational convenience in the following, we define

$$\omega = 1+2R \quad . \tag{102}$$

The magnitude of the exponential term in characteristic function (100) then can be expressed as

$$\exp\left[\frac{-\xi^2\, \omega\, N\, r^2}{1+\xi^2\omega^2}\right] , \tag{103}$$

which is monotonically decreasing for $\xi \geq 0$. Coupled with the observation immediately under (39), it is seen that $|f_h(\xi)|$ in (100) is monotonically decreasing for all $\xi \geq 0$ and any N,R,r. This property allows for a convenient termination procedure in the numerical transformation [3] of characteristic function (100). It should however be observed that (103) does not decrease to zero, but saturates at value $\exp(-Nr^2/\omega)$, regardless of how ξ increases to infinity; thus the eventual decay of the characteristic function (100) is furnished only by the leading factor.

ASYMPTOTIC BEHAVIOR OF CUMULATIVE AND EXCEEDANCE DISTRIBUTION FUNCTIONS

In appendix E, it is shown that if characteristic function (100) is substituted in (63) and (64), and the contours moved appropriately in the complex ξ-plane, then the following asymptotic behaviors obtain. The cumulative distribution function

$$P_h(u) \sim \left[\Gamma\left(\frac{N}{2}\right) 2^{N/2}(1+R)^{N/2}\right]^{-1} (-u)^{\frac{N}{2}-1} \exp\left[u - \frac{Nr^2}{2(1+R)}\right] *$$

$$* \left[1 - \frac{\frac{N}{2}-1}{u}\left(1 + \frac{N(1+2R)}{4(1+R)} + \frac{Nr^2}{4(1+R)^2}\right)\right] \quad \text{as } u \to -\infty. \tag{104}$$

We see again, in similar fashion to (61) and (62), that in order for the correction term in the second line of (104) not to be too significant, we must have $u < -N^2$. For reasons elucidated in (61) et seq., (104) is not useful for large N.

As checks on (104), we note that for r=0 and N=2, (104) reduces precisely to the upper line of (71); this latter result pertains to $\gamma=1$, N=3 and is consistent with the observation already made in the paragraph below (101). In addition, if we let r=0 and N=4 in (104), it reduces to the upper line of (75); this latter result holds for $\gamma=1$, N=5 and is likewise consistent.

Also given in appendix E are a variety of asymptotic expansions for the exceedance distribution function; the simplest one is

$$1 - P_h(u) \sim \left[2\pi^{1/2}\, 2^{N/2}\, (1+R)^{N/2} \left(Nr^2\right)^{\frac{N-1}{4}}\right]^{-1} (1+2R)^{\frac{N+1}{2}} *$$

$$* u^{\frac{N-3}{4}} \exp\left[-\frac{\left(u^{1/2} - N^{1/2}r\right)^2}{1+2R}\right] \quad \text{as } u \to +\infty; \quad r>0 . \tag{105}$$

However, the same reservations as above, regarding u large relative to N^2, are again in order.

DISTRIBUTIONS FOR N=1

If characteristic function (100) with N=1 is substituted in (63), and if the contour is moved as indicated under (66), there follows the exact result for the cumulative distribution function

$$P_h(u) = \frac{2}{\pi}\int_0^{+\infty} \frac{dt}{(1+t^2)(1+\omega(1+t^2))^{1/2}} \exp\left[u(1+t^2) - \frac{r^2(1+t^2)}{1+\omega(1+t^2)}\right] \text{ for } u \leq 0. \quad (106)$$

(This reduces to (67) for r=0, as it must.) This integral form possesses all the desirable attributes listed under (67).

If characteristic function (100) with N=1 is substituted in (64) in an attempt to get the exceedance distribution function, the analysis becomes rather difficult, due to the overlapping essential singularity and branch point of the integrand at $\xi = -i/\omega$. This problem is treated in detail in appendix F, with the result that the exceedance distribution function can be found via the characteristic function approach in terms of two integrals; see (F-21)-(F-23). However, a better numerical procedure for the exceedance distribution function is the direct result derived in (F-33); this latter integral is the one actually used here to generate the operating characteristics for γ=0, N=1.

DISTRIBUTIONS FOR N=2

The characteristic function is available from (100):

$$f_h(\xi) = (1+i\xi)^{-1} * (1-i\xi\omega)^{-1} \exp\left(\frac{i\xi 2r^2}{1-i\xi\omega}\right) \equiv f_1(\xi) * f_2(\xi) \ , \quad (107)$$

where $\omega = 1+2R$. The probability density functions corresponding to these two characteristic functions are [5, 6.631 4]

$$p_1(u) = \exp(u)\ U(-u)\ ,$$

$$p_2(u) = \frac{1}{\omega}\exp\left(-\frac{u+2r^2}{\omega}\right) I_0\left(\frac{2r}{\omega}(2u)^{1/2}\right) U(u)\ , \tag{108}$$

where U is the unit step function

$$U(u) = \begin{Bmatrix} 0 & \text{for} & u<0 \\ 1 & \text{for} & u>0 \end{Bmatrix} . \tag{109}$$

The probability density function of h is given by convolution

$$p_h(u) = \int_{-\infty}^{+\infty} dt\ p_1(t)\ p_2(u-t) \quad \text{for all } u\ . \tag{110}$$

Substitution of (108) in (110) yields

$$p_h(u) = \begin{cases} \dfrac{1}{1+\omega}\exp\left(u - \dfrac{2r^2}{1+\omega}\right) & \text{for } u \le 0 \\ \\ \dfrac{1}{1+\omega}\exp\left(u - \dfrac{2r^2}{1+\omega}\right) Q\left(\dfrac{2r}{\omega^{1/2}(1+\omega)^{1/2}}\ ,\ \left(\dfrac{2(1+\omega)u}{\omega}\right)^{1/2}\right) & \text{for } u \ge 0 \end{cases} \tag{111}$$

where the Q-function is defined in [7] and the two integrals encountered have been evaluated by use of [5, 6.631 4] and [7, (9)], respectively. The cumulative and exceedance distribution functions of h readily follow from (111), the latter by means of [7, (42)]:

$$P_h(u) = \frac{1}{1+\omega} \exp\left(u - \frac{2r^2}{1+\omega}\right) \quad \text{for } u \le 0 \; ,$$

$$1 - P_h(u) = Q\left(\frac{2r}{\omega^{1/2}} \, , \left(\frac{2u}{\omega}\right)^{1/2}\right) -$$

$$- \frac{1}{1+\omega} \exp\left(u - \frac{2r^2}{1+\omega}\right) Q\left(\frac{2r}{\omega^{1/2}(1+\omega)^{1/2}} \, , \left(\frac{2(1+\omega)u}{\omega}\right)^{1/2}\right) \quad \text{for } u \ge 0. \qquad (112)$$

(As a check, for r=0, then (111) and (112) reduce to (70) and (71) respectively, as they must.)

DISTRIBUTIONS FOR N=4

The characteristic function is available from (100):

$$f_h(\xi) = (1+i\xi)^{-2} * (1-i\xi\omega)^{-2} \exp\left(\frac{i\xi 4r^2}{1-i\xi\omega}\right) \equiv f_1(\xi) * f_2(\xi) \; , \qquad (113)$$

where $\omega = 1+2R$. The probability density functions corresponding to these two characteristic functions are [5, 6.631 4]

$$p_1(u) = -u \exp(u) \, U(-u) \; ,$$

$$p_2(u) = \frac{1}{2\omega r} \exp\left(-\frac{u+4r^2}{\omega}\right) u^{1/2} \, I_1\left(4ru^{1/2}/\omega\right) U(u) \; . \qquad (114)$$

The probability density function of h is given by convolution (110). In preparation for that result, we use the shorthand notation

$$q_M(u) = Q_M\left(2r\left(\frac{2}{\omega(1+\omega)}\right)^{1/2} \, , \quad \left(\frac{2u(1+\omega)}{\omega}\right)^{1/2}\right) \, ,$$

$$\tilde{q}_M(u) = Q_M\left(2r\left(\frac{2}{\omega}\right)^{1/2} \, , \left(\frac{2u}{\omega}\right)^{1/2}\right), \qquad (115)$$

where the Q_M-function is defined in [8]. We also present a new integral result that will be needed in the sequel,

$$\int_c^{+\infty} dx\ x\ Q_M(b,\ ax) = \frac{1}{2a^2}[b^2 Q_{M+1}(b,ac)+2M\ Q_M(b,ac)-a^2c^2 Q_{M-1}(b,ac)], \quad (116)$$

which can be interpreted as the limit of [8, (31)] as $p \to 0+$. Substitution of (114) in (110) yields probability density function

$$p_h(u) = \begin{cases} \dfrac{1}{(1+\omega)^4} \exp\left(u - \dfrac{4r^2}{1+\omega}\right)\left[4r^2+2\omega(1+\omega)-(1+\omega)^2 u\right] & \text{for } u \le 0 \\ \\ \dfrac{1}{(1+\omega)^4} \exp\left(u - \dfrac{4r^2}{1+\omega}\right)\left[4r^2 q_3(u)+2\omega(1+\omega)q_2(u)-(1+\omega)^2 u q_1(u)\right] & \text{for } u \ge 0 \end{cases} \quad (117)$$

where (115) has been used; the upper line employed [5, 6.631 4], while the lower line used (116) and an integration by parts procedure to be elaborated below in the exceedance distribution function evaluation.

The cumulative distribution function for $u \le 0$ follows readily by integration of (117):

$$P_h(u) = \frac{1}{(1+\omega)^4} \exp\left(u - \frac{4r^2}{1+\omega}\right)\left[4r^2+(1+\omega)(1+3\omega)-(1+\omega)^2 u\right] \quad \text{for } u \le 0. \quad (118)$$

For $u \ge 0$, a modified approach is required. Integration of (110) yields cumulative distribution function

$$P_h(u) = \int_{-\infty}^{+\infty} dt\ p_2(t)\ P_1(u-t)\ . \quad (119)$$

Probability density functions p_1 and p_2 are available in (114), and cumulative distribution function P_1 follows as

$$P_1(u) = \begin{Bmatrix} (1-u)\exp(u) & \text{for } u \le 0 \\ 1 & \text{for } u \ge 0 \end{Bmatrix} . \tag{120}$$

Substitution of (114) and (120) in (119) yields, for $u \ge 0$,

$$P_h(u) = \int_0^u dt \, \frac{1}{2\omega r} \exp\left(-\frac{t+4r^2}{\omega}\right) t^{1/2} \, I_1\left(4rt^{1/2}/\omega\right) +$$

$$+ \int_u^{+\infty} dt \, \frac{1}{2\omega r} \exp\left(-\frac{t+4r^2}{\omega}\right) t^{1/2} \, I_1\left(4rt^{1/2}/\omega\right) (1-u+t) \exp(u-t) =$$

$$= 1-\tilde{q}_2(u) + \frac{1}{2\omega r} \exp\left(u-\frac{4r^2}{\omega}\right) \int_u^{+\infty} dt \, t^{1/2} \exp\left(-t \, \frac{1+\omega}{\omega}\right) I_1\left(4rt^{1/2}/\omega\right) (1-u+t), \tag{121}$$

via [8, (22)]. We now integrate by parts, letting $U(t)=1-u+t$, and the remainder $dV(t)$. Then using [8, (22)] again, we find

$$V(t) = -\frac{2\omega r}{(1+\omega)^2} \exp\left(\frac{4r^2}{\omega(1+\omega)}\right) q_2(t) . \tag{122}$$

Combining these results, (121) and (116) yield

$$1 - P_h(u) = \tilde{q}_2(u) - \frac{1}{(1+\omega)^2} \exp\left(u - \frac{4r^2}{1+\omega}\right) q_2(u) - \int_u^{+\infty} dt \, q_2(t) =$$

$$= \tilde{q}_2(u) - \frac{1}{(1+\omega)^4} \exp\left(u - \frac{4r^2}{1+\omega}\right)\left[4r^2 q_3(u) + (1+\omega)(1+3\omega) q_2(u) - (1+\omega)^2 u q_1(u)\right]$$

$$\text{for } u \ge 0. \tag{123}$$

The final results for N=4 are given by (115), (118), and (123). This case was used as a numerical check on the computational approach [3] proceeding directly from characteristic function (100) to the exceedance distribution function, with excellent agreement for numerous values of R, r, and u.

GRAPHICAL RESULTS FOR $\gamma=0$, SAMPLE MEAN NOT REMOVED

SUMMARY OF PARTICULAR CASE CONSIDERED

The situation of interest has already been summarized in (76)-(78); in addition, we have a common value for the normalized means,

$$\frac{\mu_u}{D_u^{1/2}} = \frac{\mu_v}{D_v^{1/2}} = r \ , \tag{124}$$

and the normalized crosscorrelator output is not (79)-(80), but rather is, for $\gamma=0$,

$$h = \frac{1}{(D_u D_v)^{1/2}} \sum_{n=1}^{N} u_n v_n \ . \tag{125}$$

The characteristic function is given by (100):

$$f_h(\xi) = [(1+i\xi)(1-i\xi\omega)]^{-N/2} \exp\left(\frac{i\xi N r^2}{1-i\xi\omega}\right), \tag{126}$$

where $\omega = 1+2R$. When the signal is absent, then R=0; however the false alarm probability corresponding to this characteristic function still depends on r. Thus since the sample mean has not been removed, the operating characteristics will also depend on r. Since results for r=0 can be found from an earlier section, we only consider r>0 here.

OPERATING CHARACTERISTICS FOR $\gamma=0$

A sample program for evaluation of the cumulative and exceedance distribution functions corresponding to characteristic function (126), and thereby the detection probability vs. false alarm probability operating characteristics of the crosscorrelator without sample mean removal, is given in appendix G. In figures* 15-35 are presented the operating characteristics for values of

*See page 65 et seq.

$$N = 1,\ 2,\ 3,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128,\ 256, \tag{127}$$

and for various values of r. The case of N=1 was accomplished by use of (106) and (F-33); results for N=2 employed (112); and the remainder for $N \geq 3$ employed a numerical procedure [3] proceeding directly from characteristic function (126) to the exceedance distribution function. False alarm probabilities P_F in the range 1E-10 to .5 and detection probabilities P_D covering 1E-10 to .999 are presented. The abscissa and ordinate on these plots employ a normal probability transformation, as explained earlier in (84)-(86).

Values of signal-to-noise ratio R are taken as $R=2^n$, where n assumes values appropriate for each plot in order to cover the full range of probabilities of interest. Values of normalized mean r in (124) have been taken as r=1 and 2, with the exception of figure 19 where one example for r=4 was added.

Without exception, increasing r from zero degrades the operating characteristics of the crosscorrelator. For example, figures 17, 18, 19 give a succession of operating characteristics for r = 1, 2, 4 respectively, and for common values of signal-to-noise ratio R. (In order to determine the operating characteristics for r=0 here, we can look at the earlier results in figures 1-14, but for a value of N which is one greater there.) Thus, not removing the sample mean from the crosscorrelator output requires a larger threshold setting for a specified false alarm probability and thereby lowers the detection probability and degrades performance.

GAUSSIAN APPROXIMATION

If the crosscorrelator output is Gaussian, for both signal absent as well as present, the earlier derivations in (83)-(86) pertain. Now reference to (101) yields statistics

$$m_0 = Nr^2, \quad m_1 = N(R+r^2)\ ,$$

$$\sigma_0^2 = N(1+2r^2), \quad \sigma_1^2 = N[1+2R+2R^2+2(1+2R)r^2], \tag{128}$$

since setting signal-to-noise ratio R=0 corresponds to hypothesis 0, signal absent. Substitution in (86) yields the normal probability approximation

$$y = \frac{N^{1/2}\, R + (1+2r^2)^{1/2}\, x}{[1+2R+2R^2+2(1+2R)r^2]^{1/2}} \, . \tag{129}$$

These straight lines are superposed as dashed lines in figures 32-35 for N=128 and 256. The Gaussian approximation is moderately good for large N such as 256, and in fact crosses the exact curves (solid) at a point which is labeled with the corresponding value of n in signal-to-noise ratio $R=2^n$.

An obvious shortcoming of the Gaussian approximation (129) is apparent:

$$\lim_{R\to\infty} y = \left(\frac{N}{2}\right)^{1/2} \quad \text{for any } x, r \, . \tag{130}$$

Reference to (85) then yields the interpretation

$$\lim_{R\to\infty} P_D = \Phi\left(\left(\frac{N}{2}\right)^{1/2}\right) < 1 \quad \text{for any } P_F, r \, . \tag{131}$$

That is, as input signal-to-noise ratio R tends to infinity, the approximate detection probability saturates at a value less than 1, regardless of the false alarm probability and normalized mean r. Thus the Gaussian approximation must certainly be inaccurate for large R; the exact discrepancy for probabilities of practical interest is displayed in figures 32-35.

SUMMARY

A closed form expression for the characteristic function of the output of a crosscorrelator, with or without sample mean removal, has been derived in (23)-(24) for general values of: the number of terms added to yield the correlator output, the means and variances in each of the two input channels, the degree of correlation between the two channels, and the scale factor employed in the sample mean removal. A program for the evaluation of the cumulative and exceedance distribution functions of this general case has also been presented. These results can furnish the basis of a study of the error probabilities of a correlator required to decide between alternative hypotheses on the input statistics [1]; this problem will in fact be the subject of a future technical report by this author.

The general results were first specialized to a signal and noise model, and then to the two distinct cases of sample mean removal ($\gamma=1$) and no sample mean removal ($\gamma=0$). Plots of the operating characteristics for numerous values of N and signal-to-noise ratio R were then displayed for a wide range of detection probability vs false alarm probability. Some new analytic results for cumulative and exceedance distribution functions, especially for small N, were derived and used as checks on the general numerical procedure. Comparisons with a Gaussian approximation indicated quantitatively when that simplification is valid. Asymptotic results derived were useful for small N, but not for large N except in the region of probabilities too small to be of practical importance.

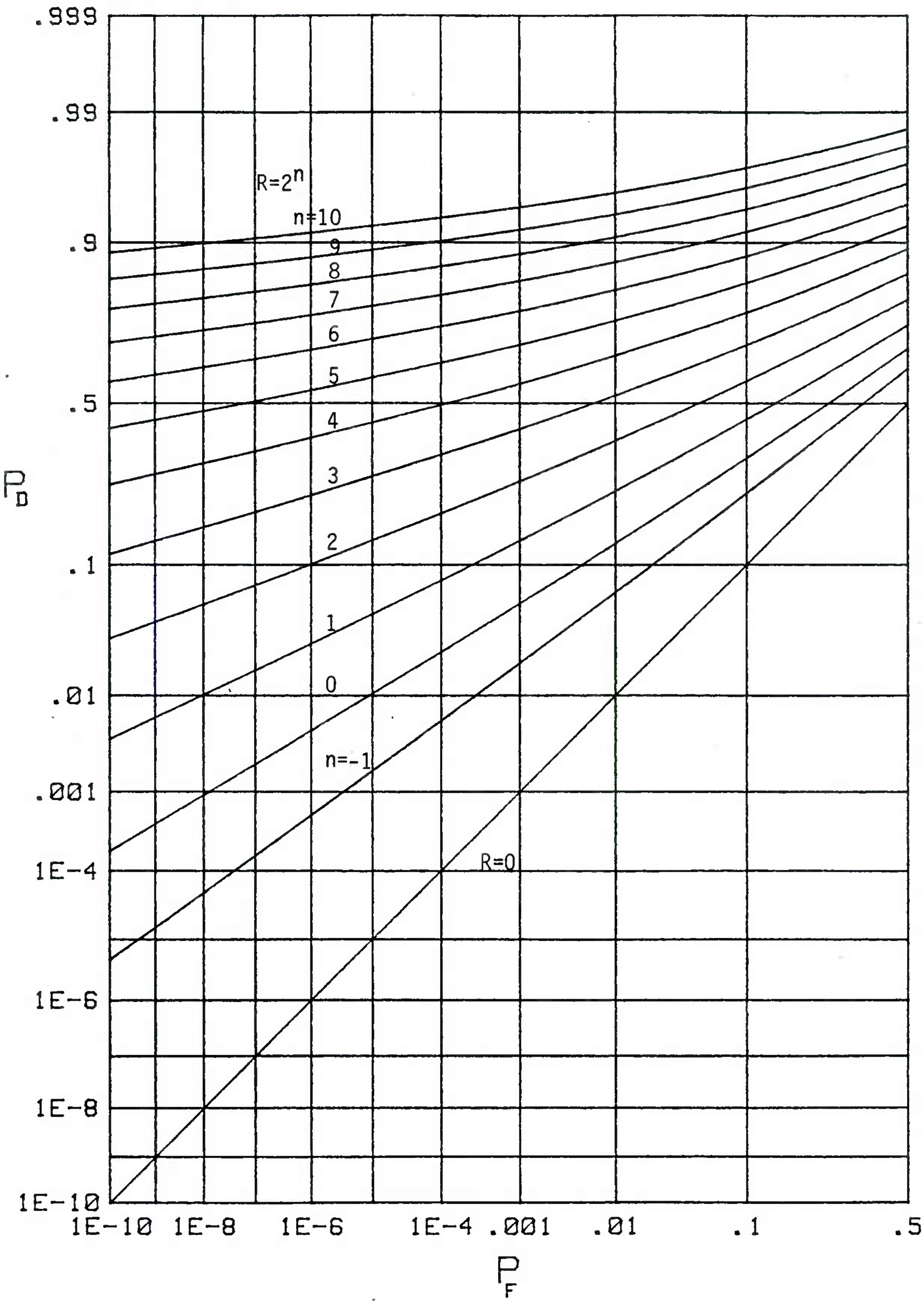

Figure 1. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 2

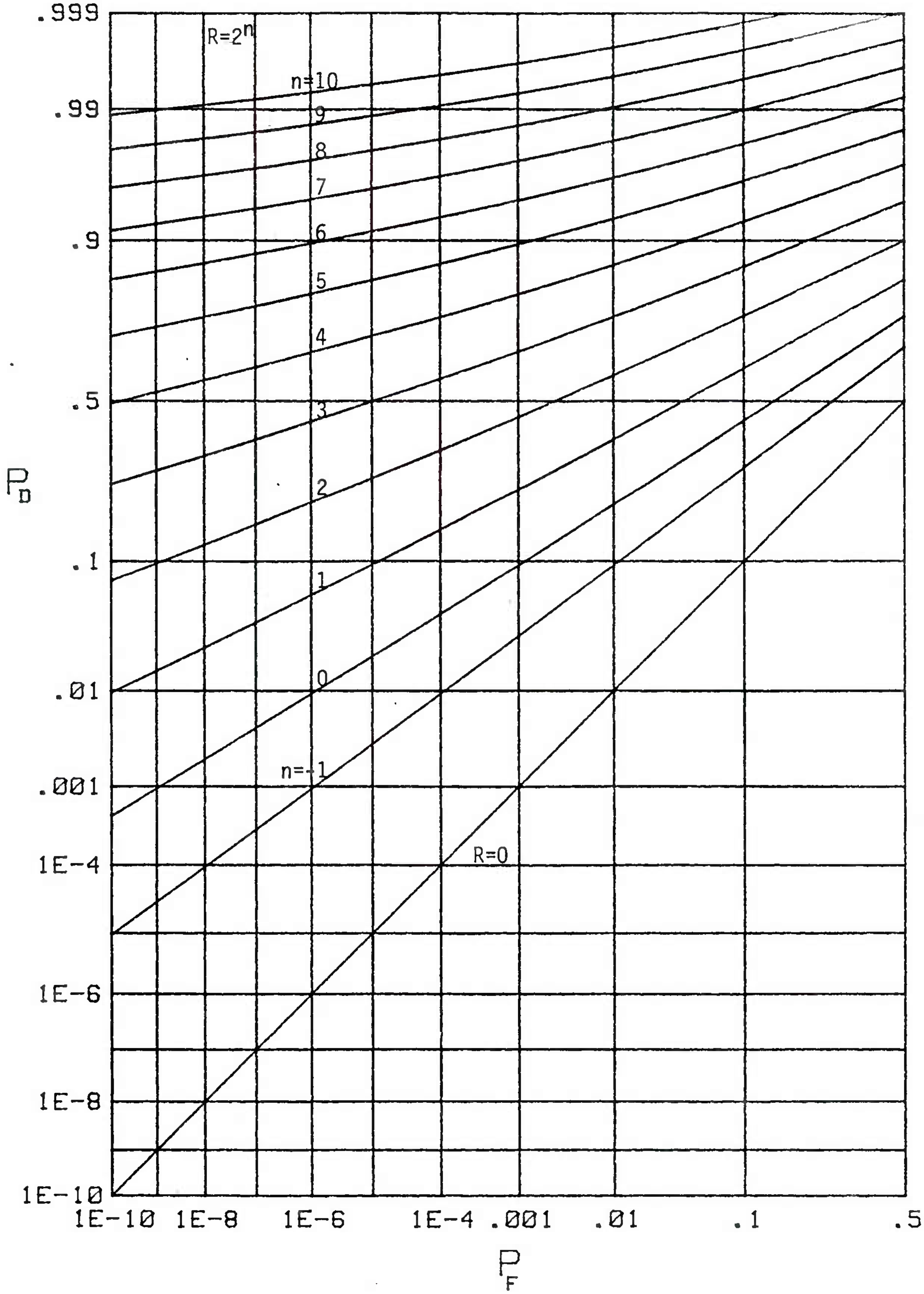

Figure 2. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 3

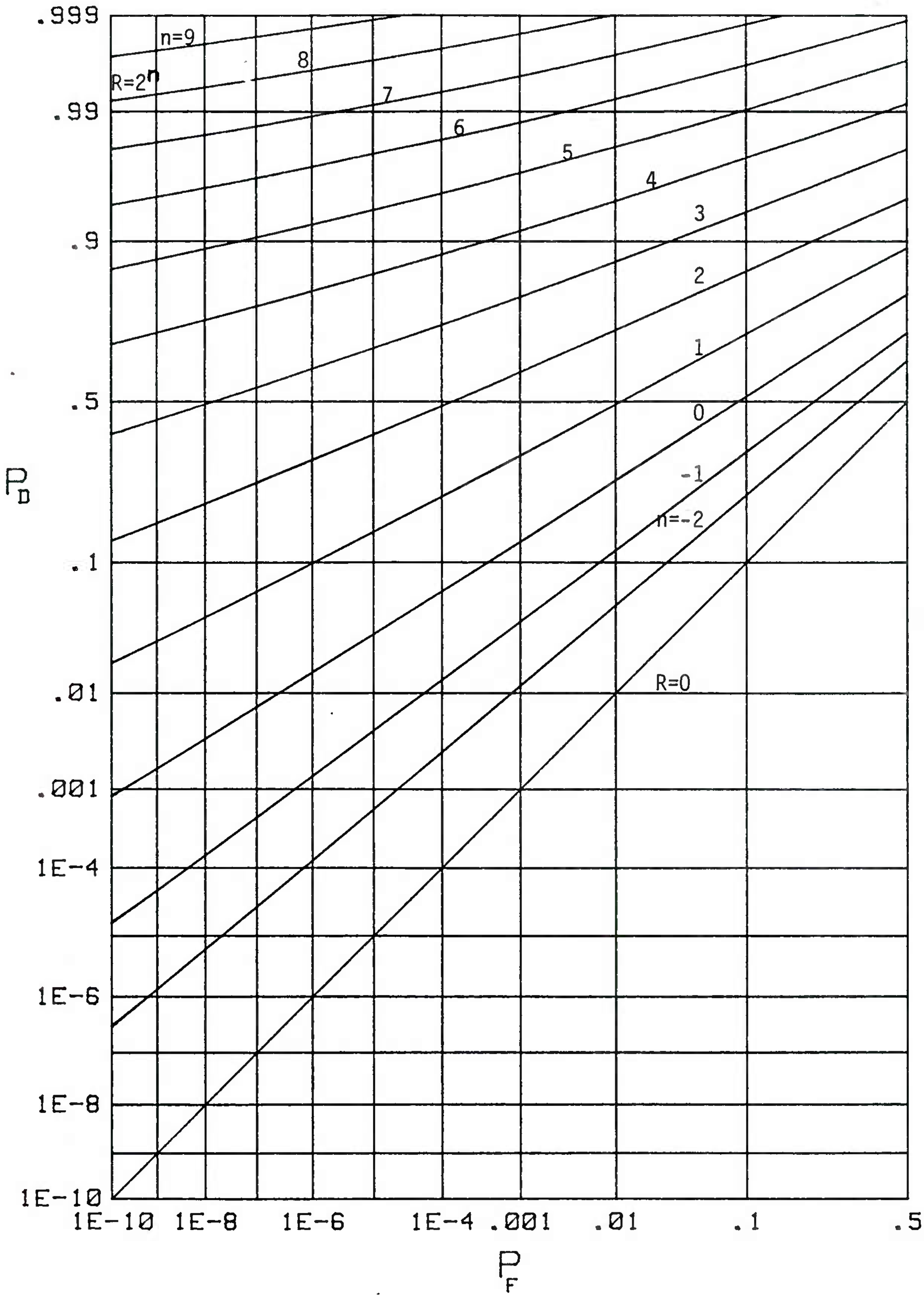

Figure 3. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 4

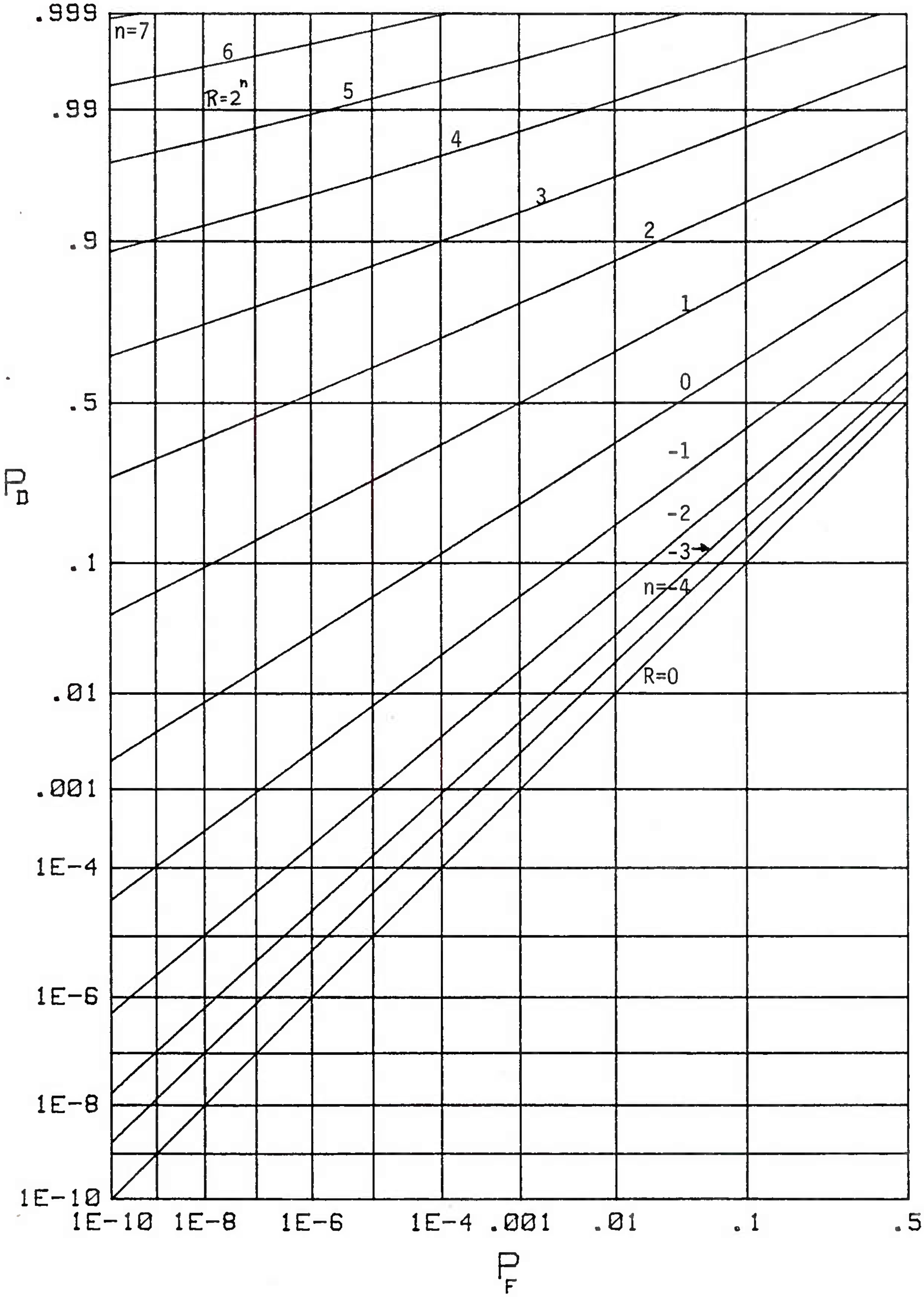

Figure 4. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 6

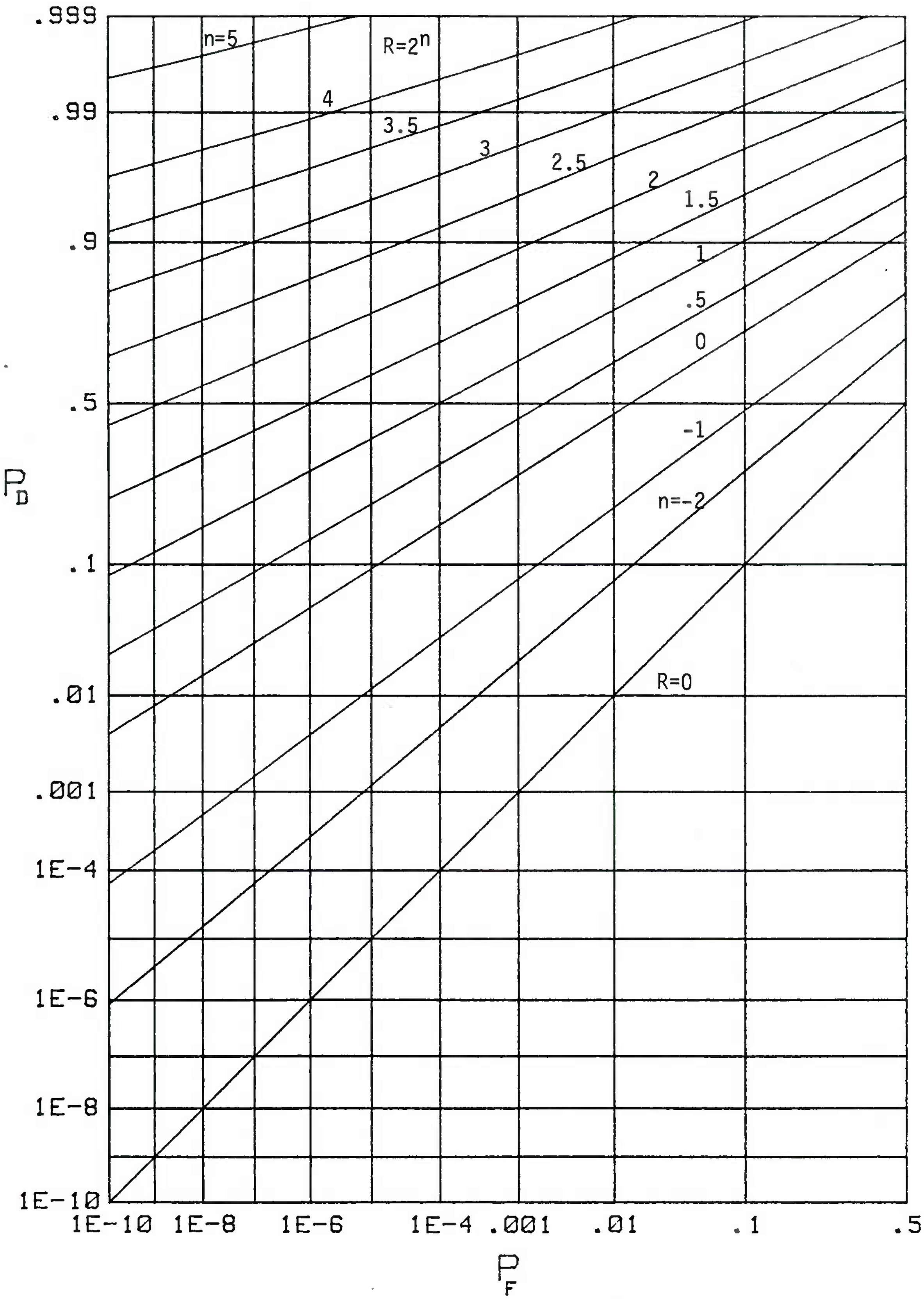

Figure 5. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 8

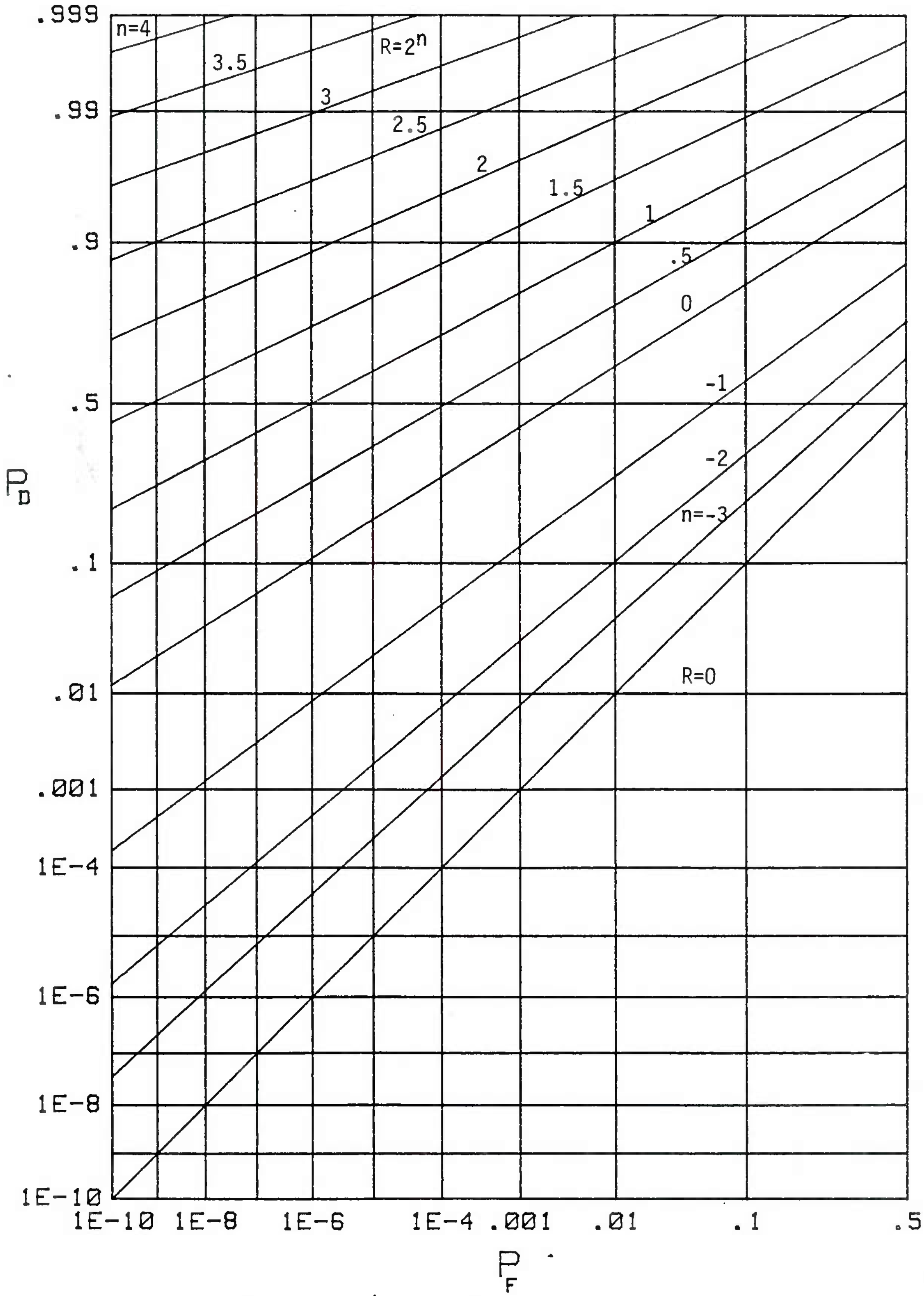

Figure 6. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 12

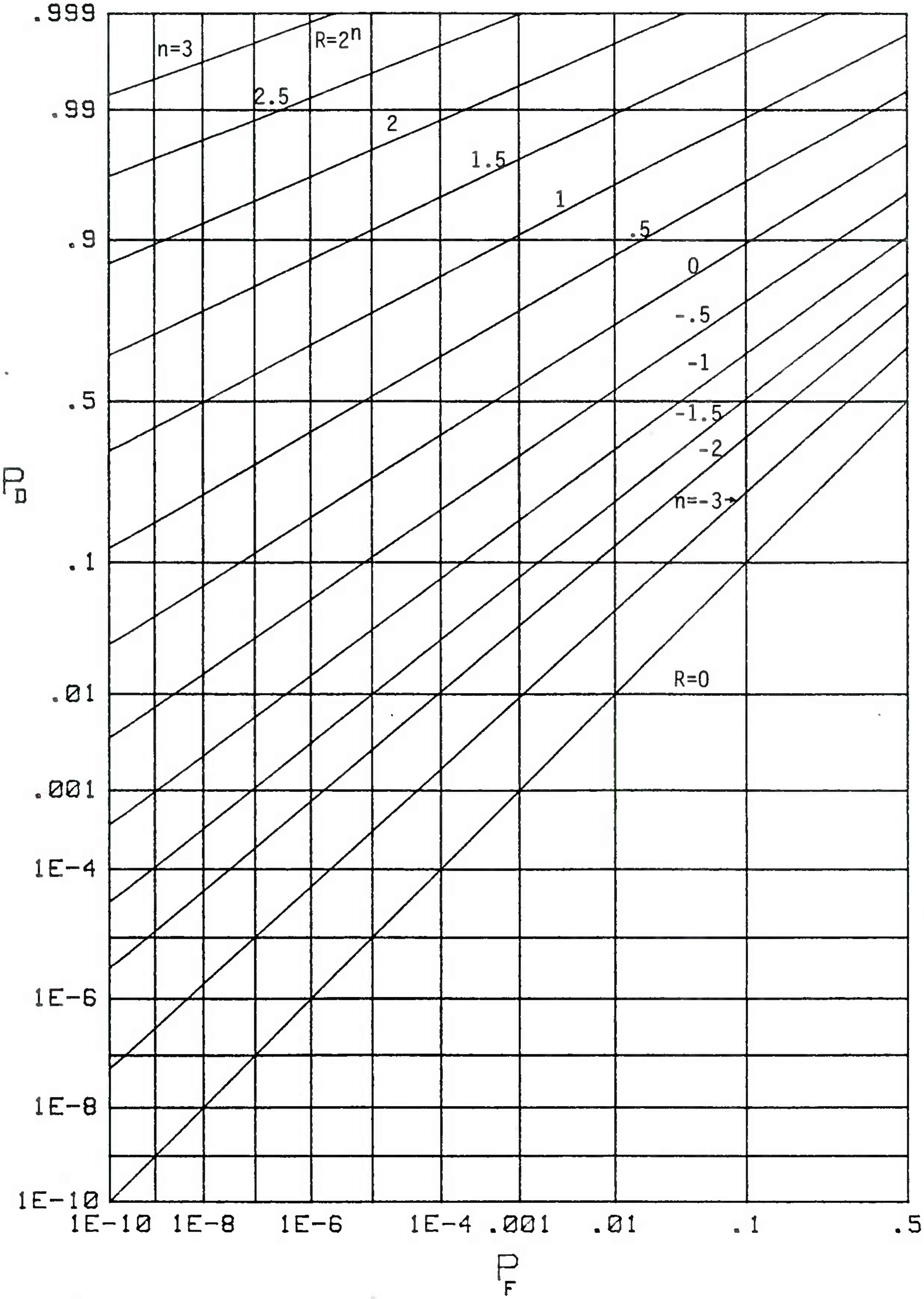

Figure 7. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 16

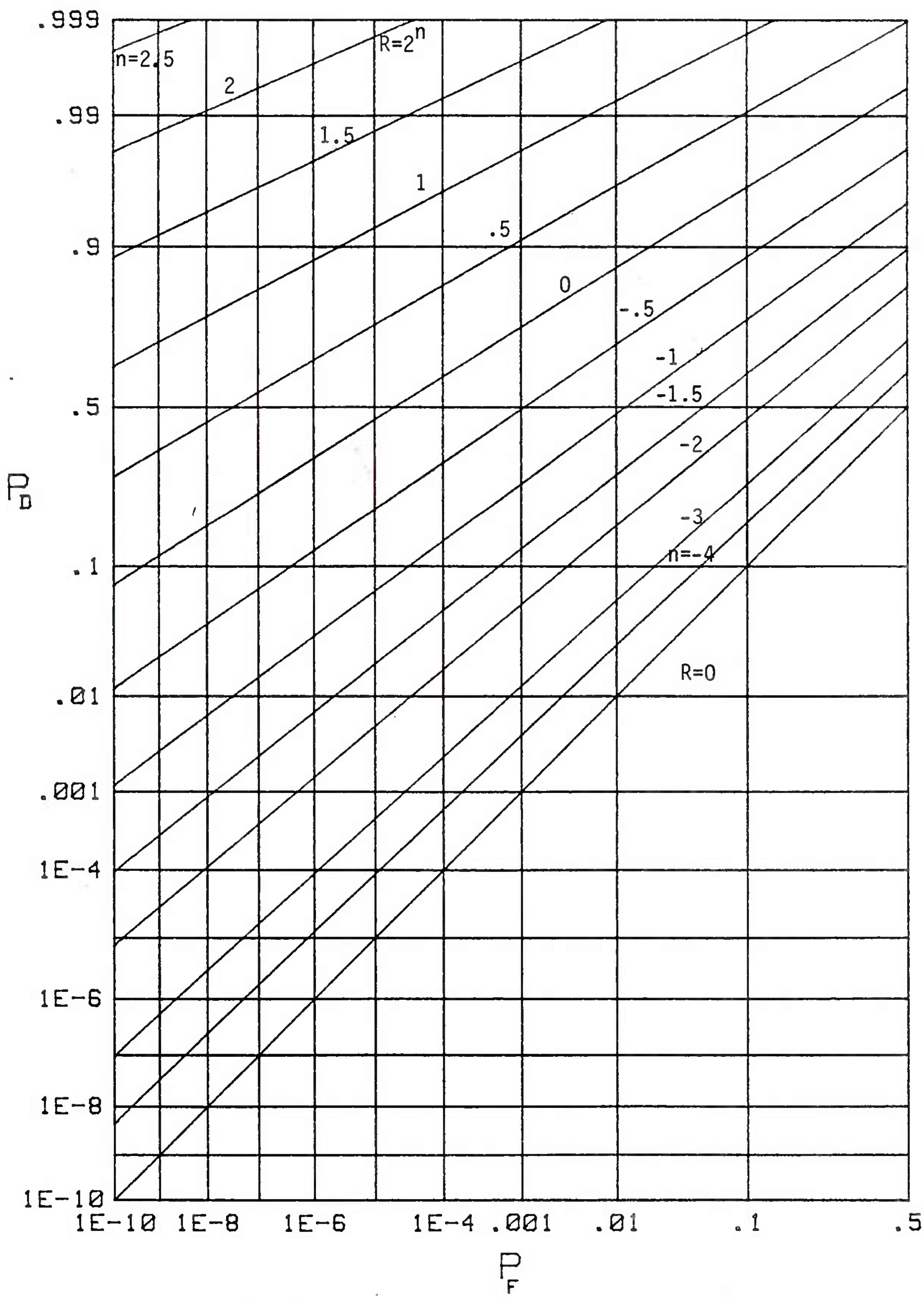

Figure 8. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 24

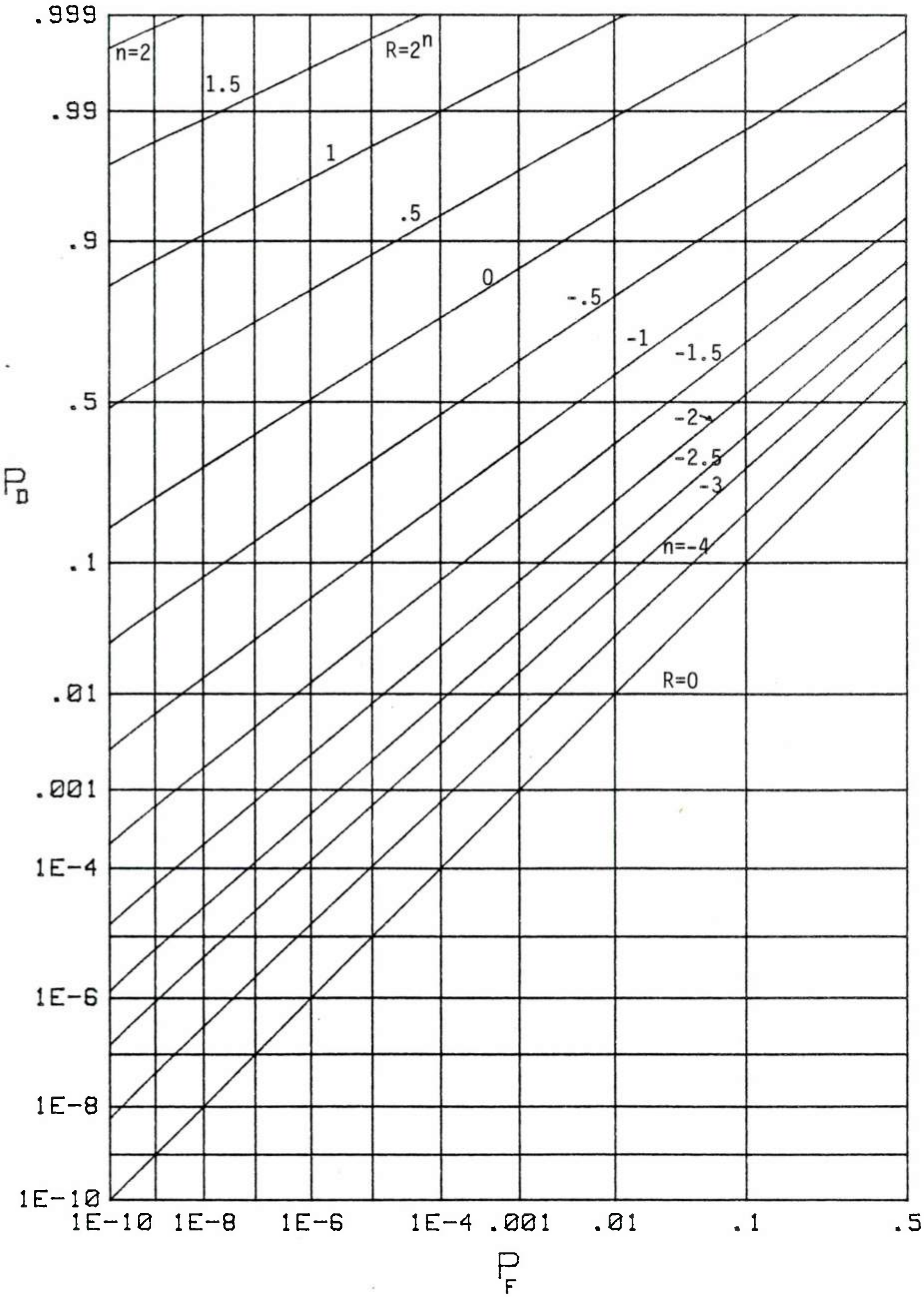

Figure 9. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 32

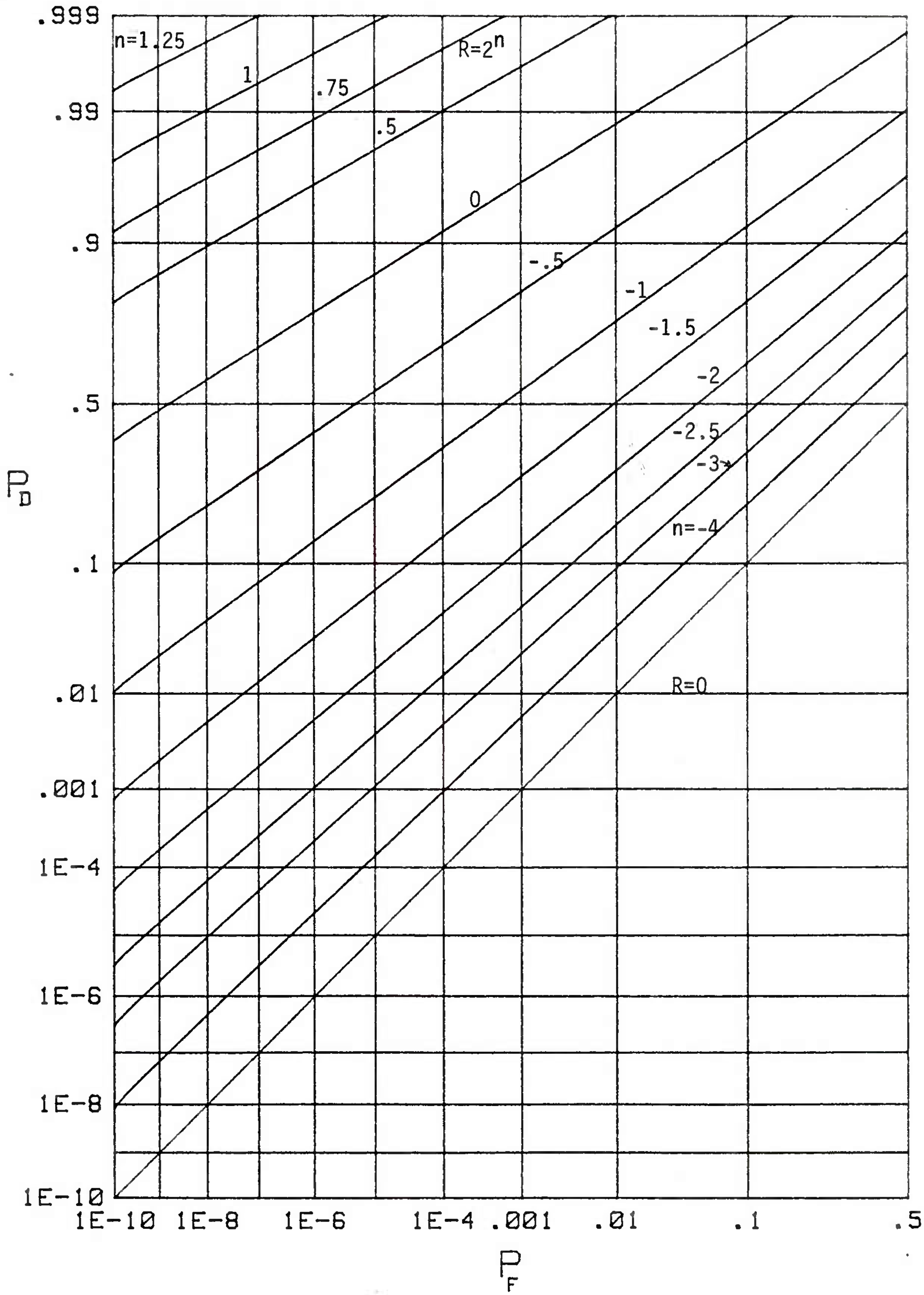

Figure 10. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 48

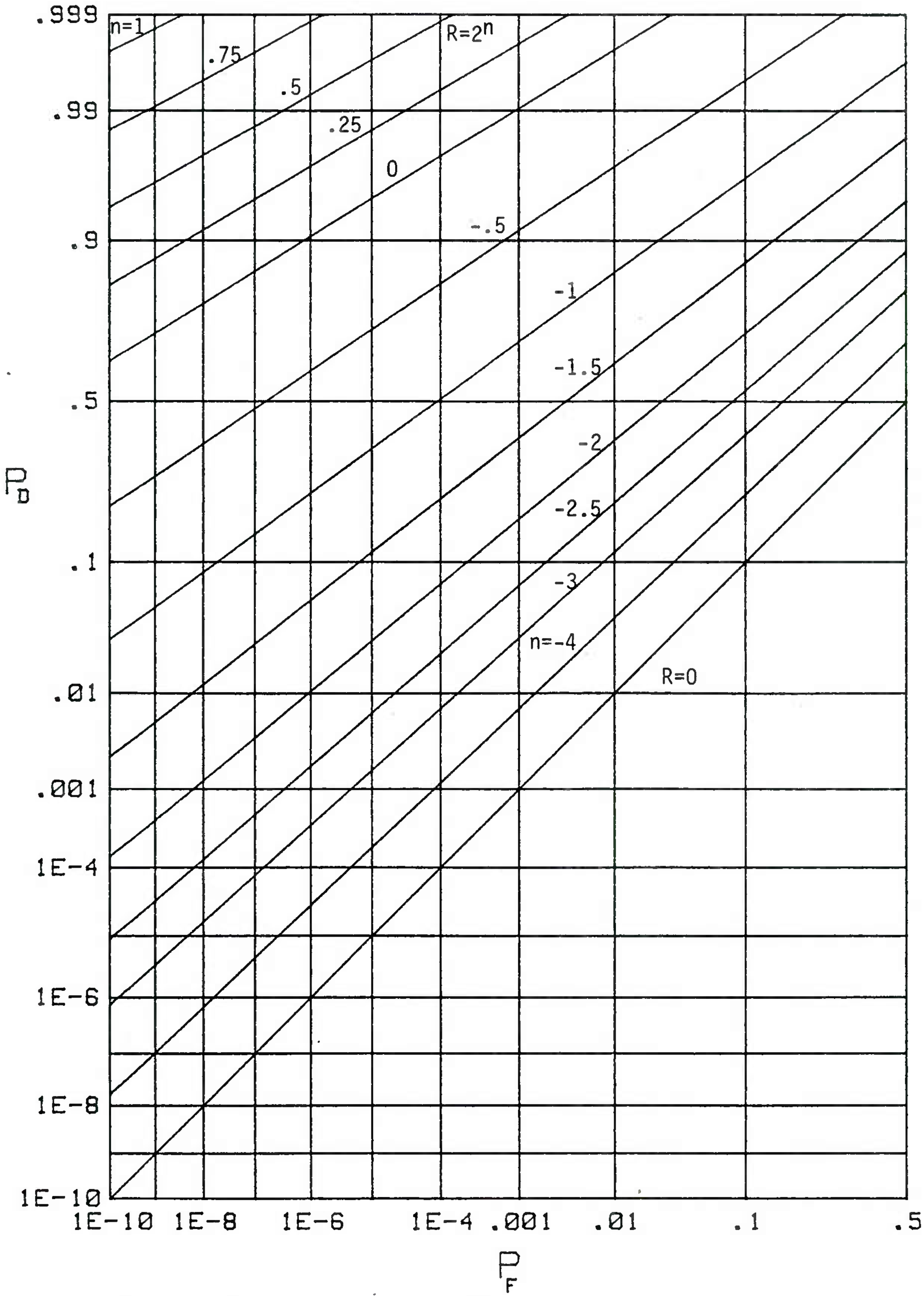

Figure 11. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 64

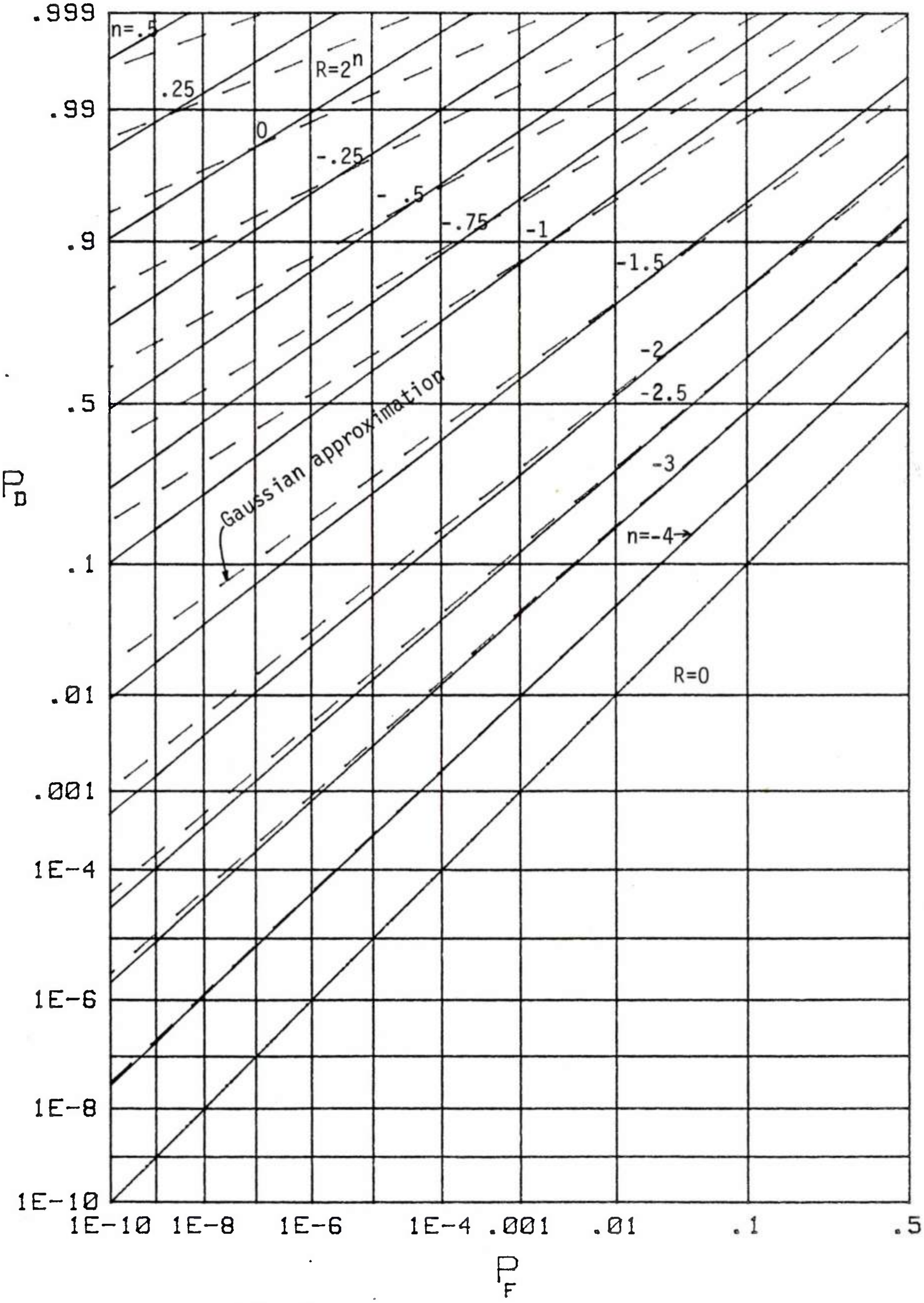

Figure 12. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 96

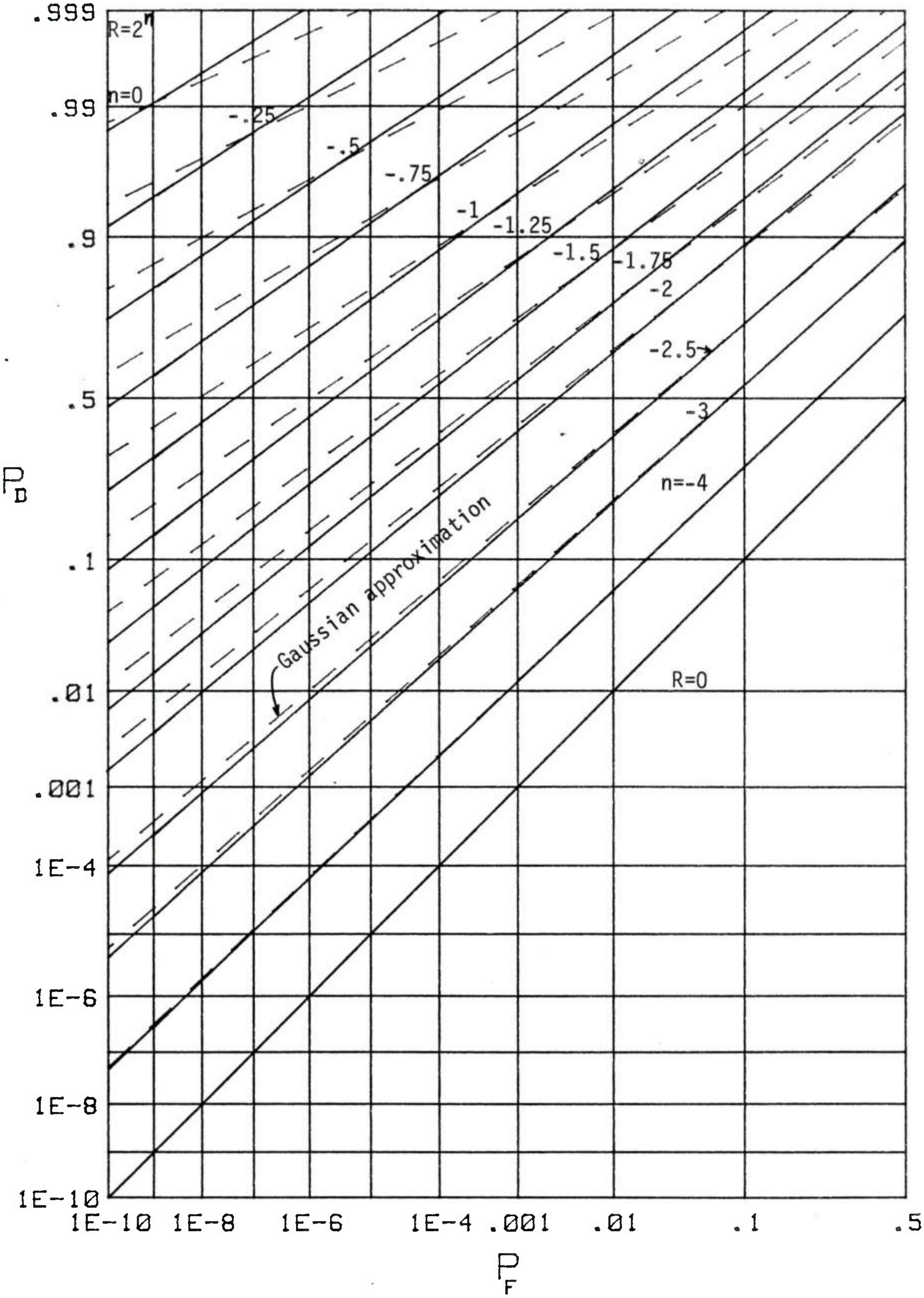

Figure 13. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 128

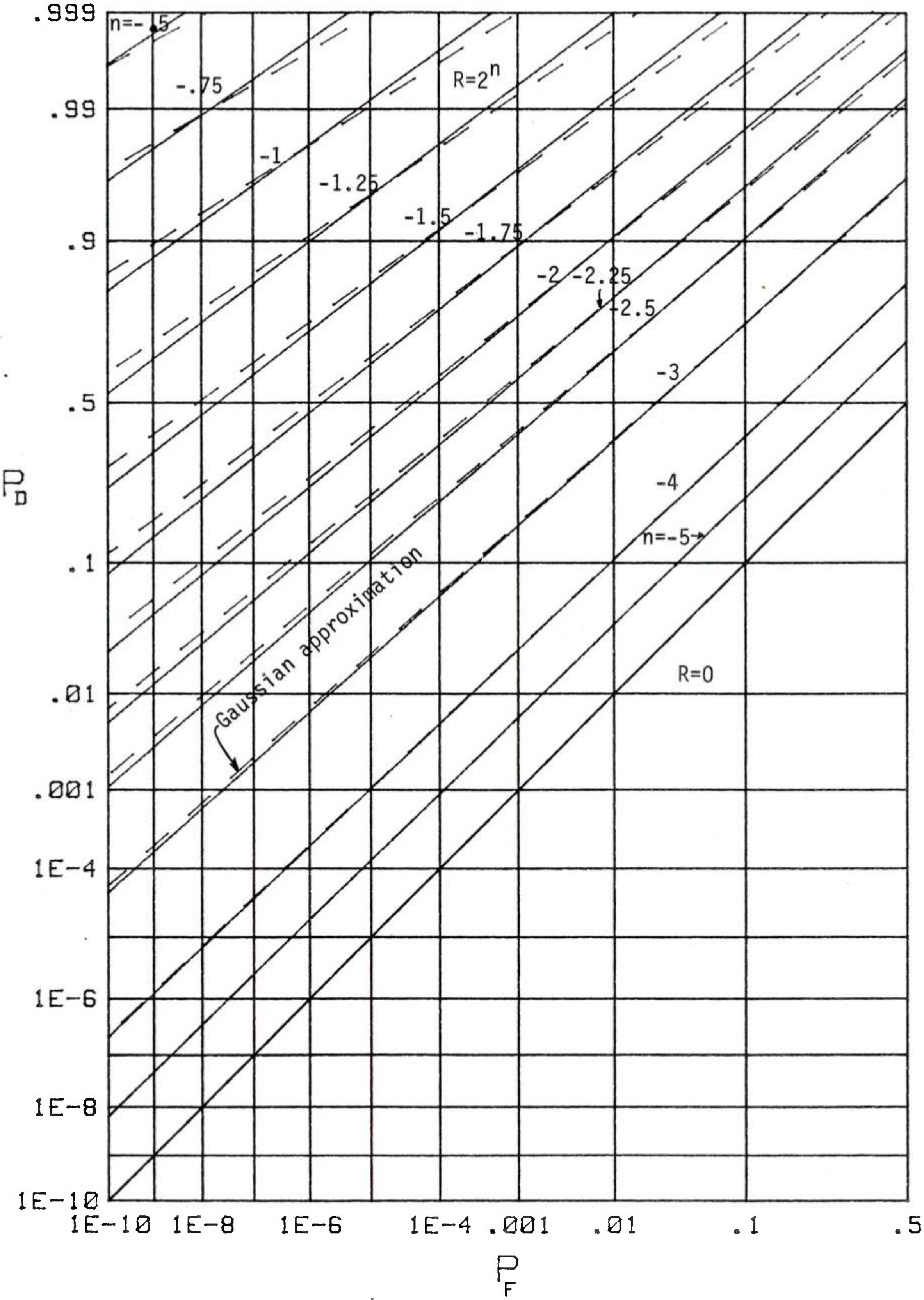

Figure 14. Operating Characteristics for Cross-Correlator with Sample Mean Removal, N = 256

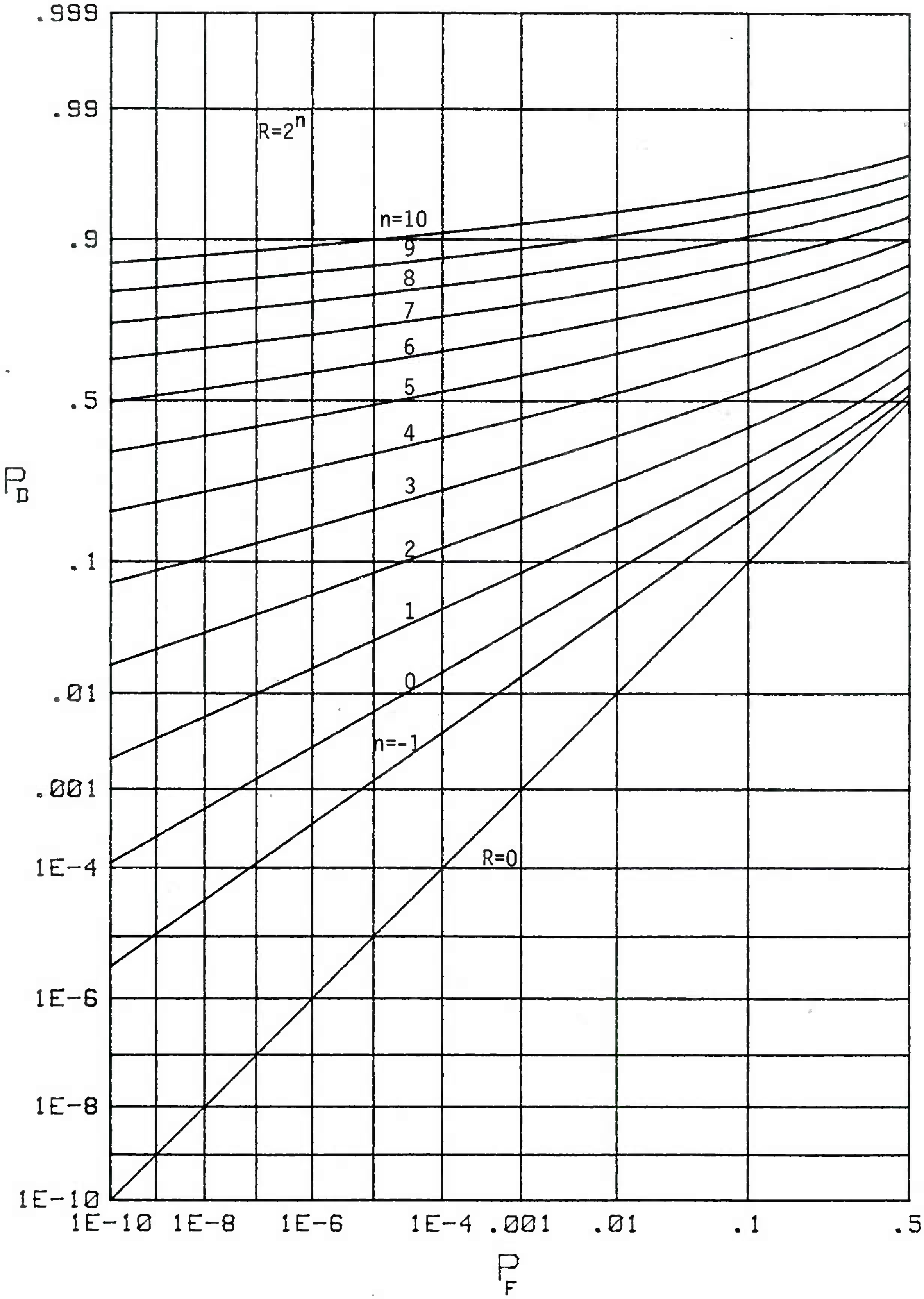

Figure 15. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=1, r=1

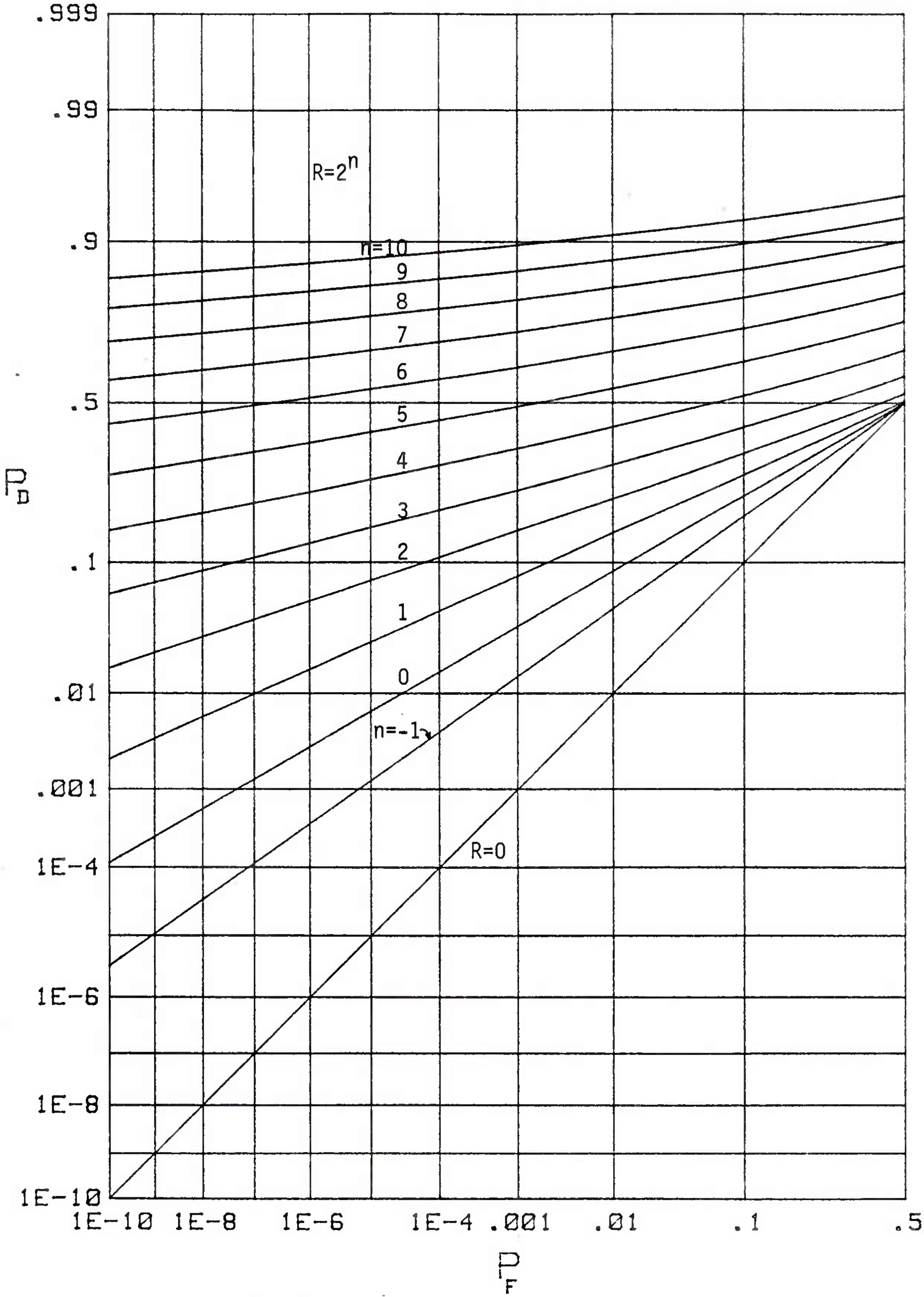

Figure 16. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=1, r=2

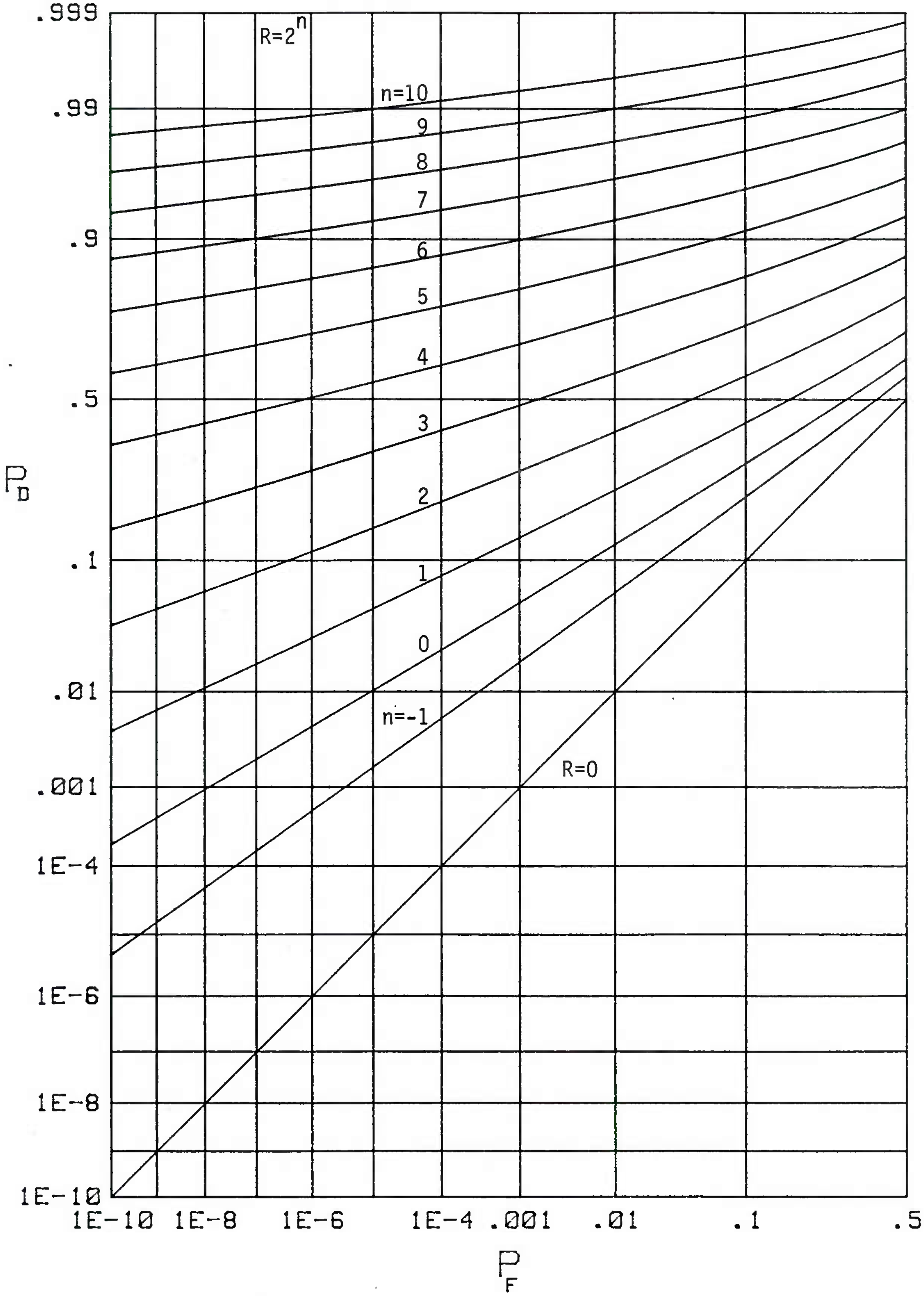

Figure 17. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=2, r=1

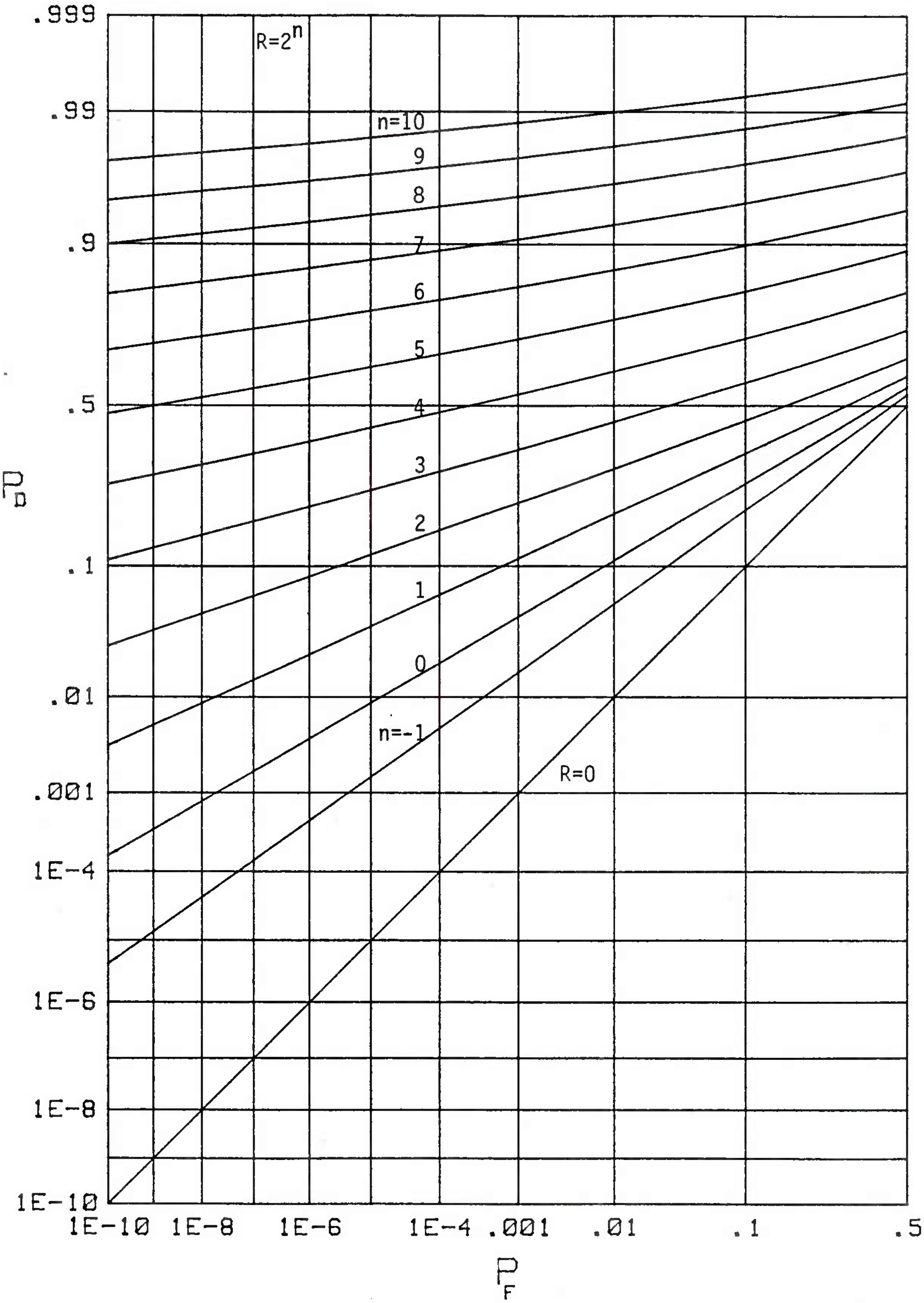

Figure 18. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=2, r=2

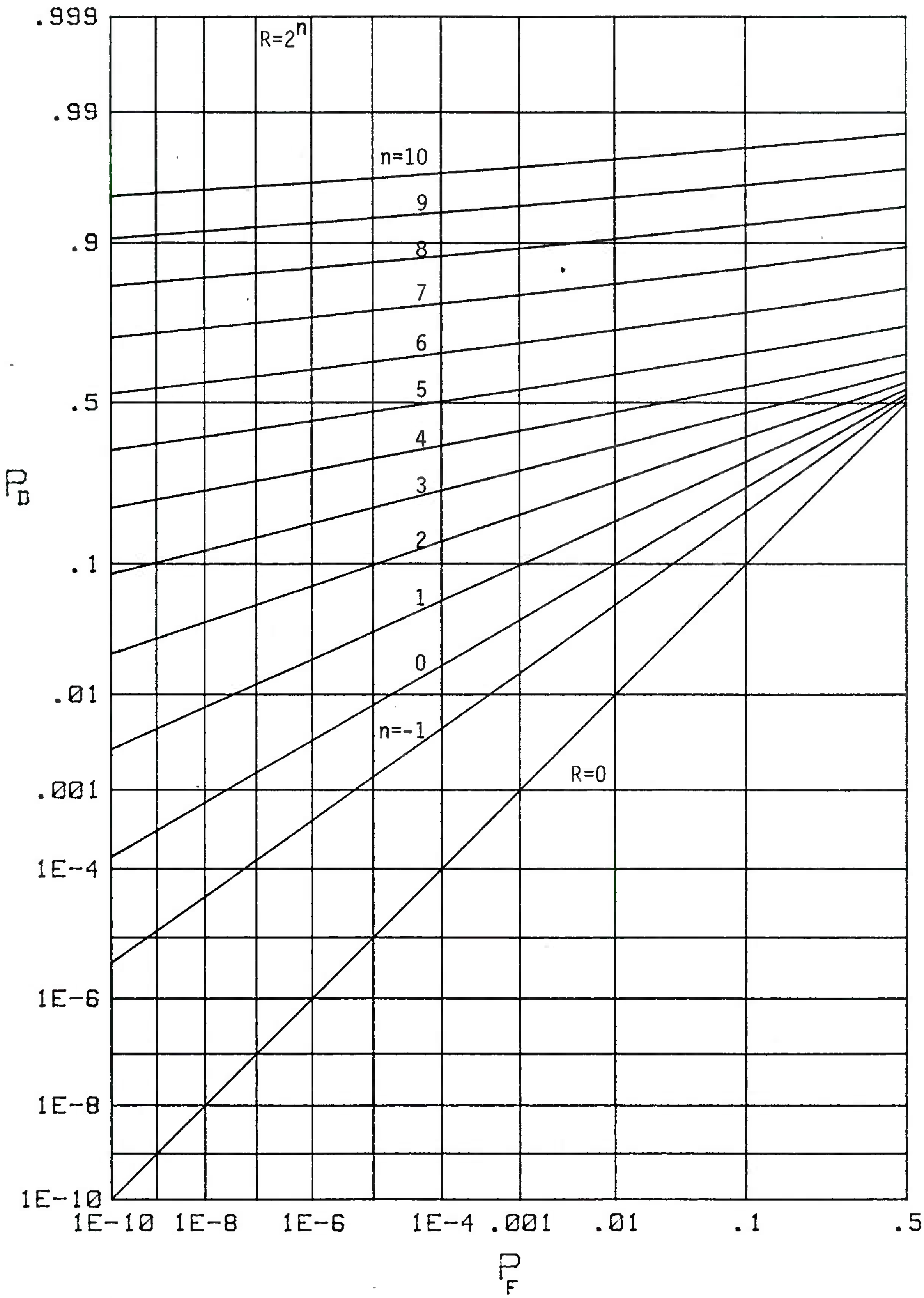

Figure 19. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=2, r=4

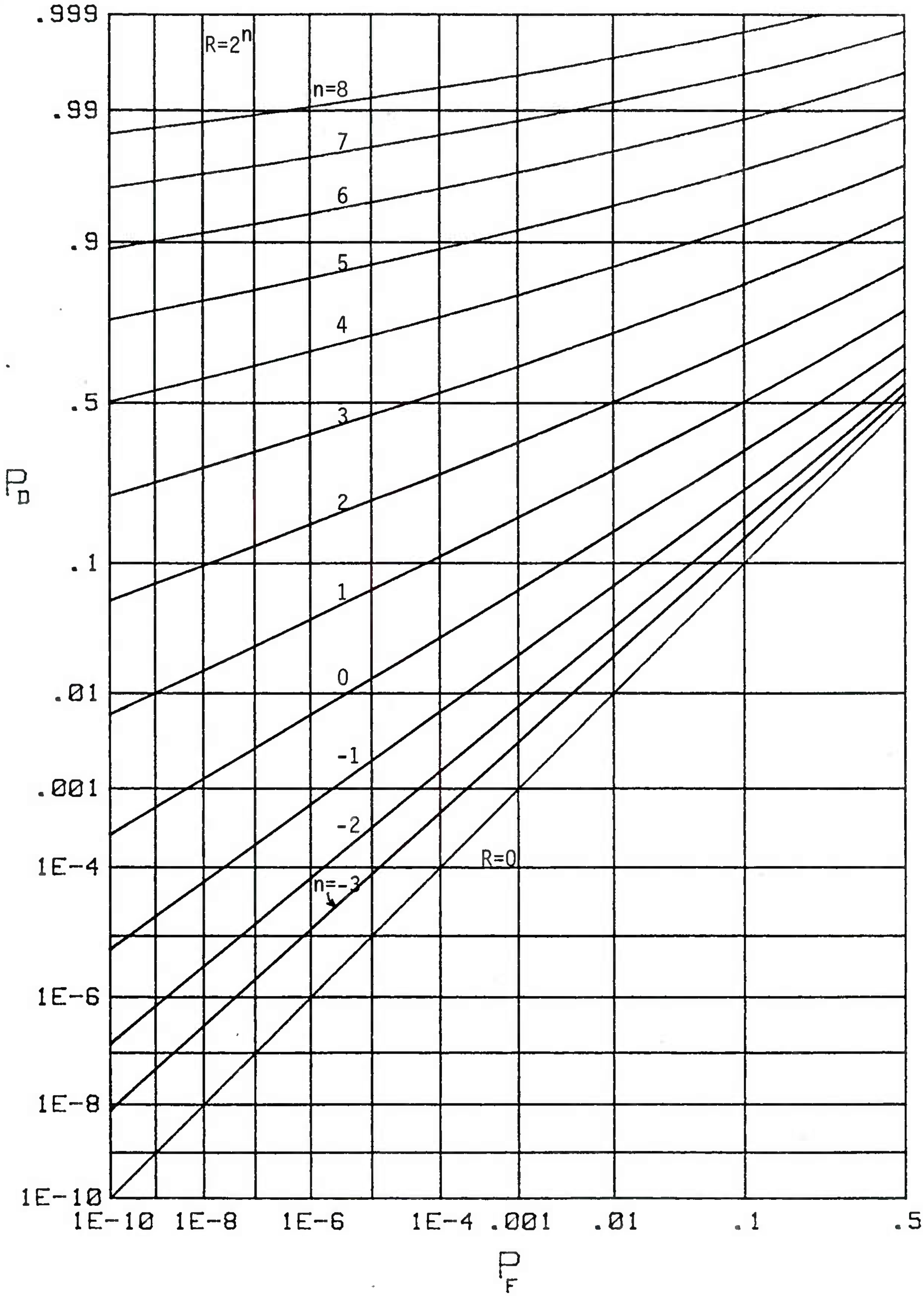

Figure 20. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=3, r=1

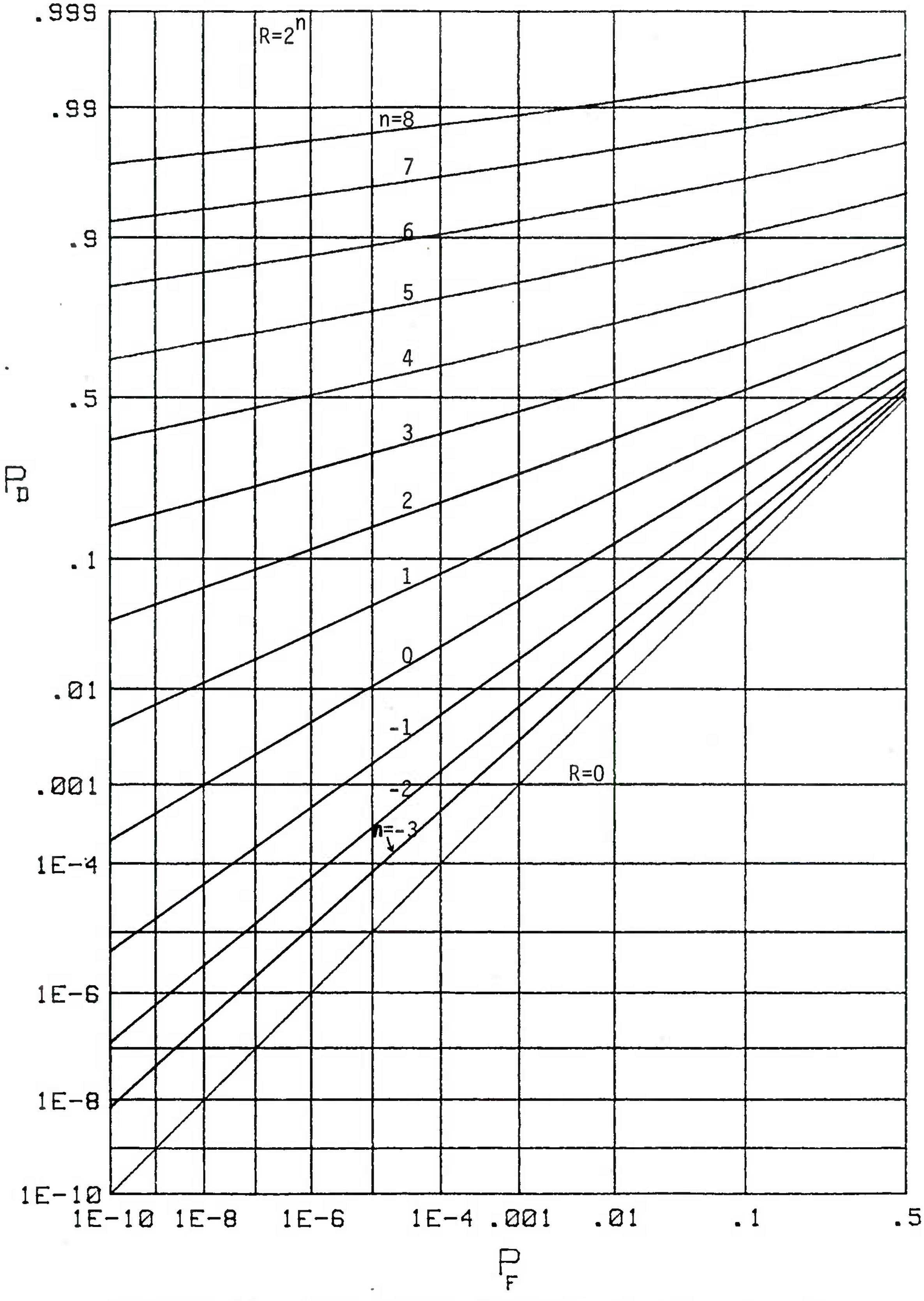

Figure 21. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=3, r=2

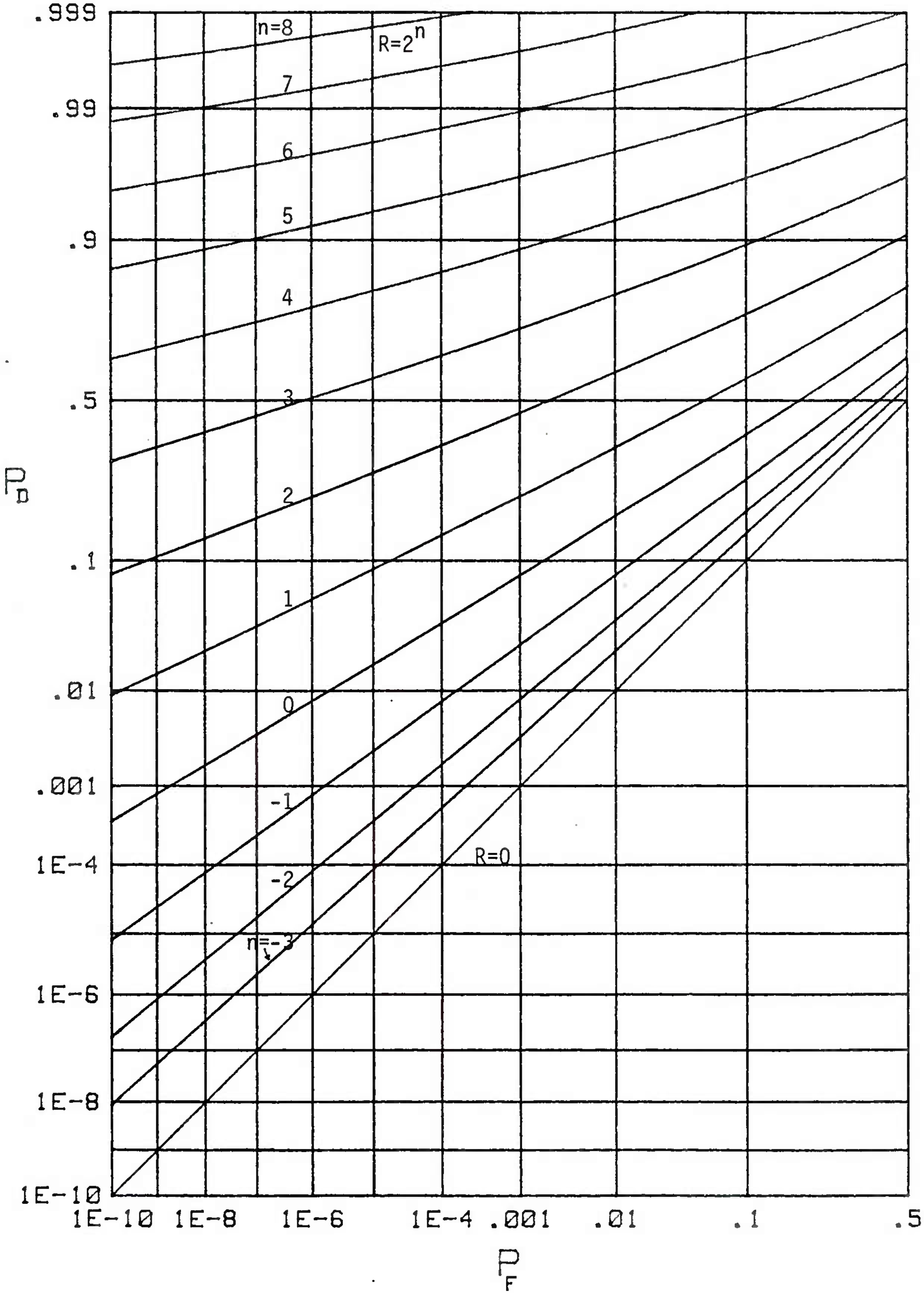

Figure 22. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=4, r=1

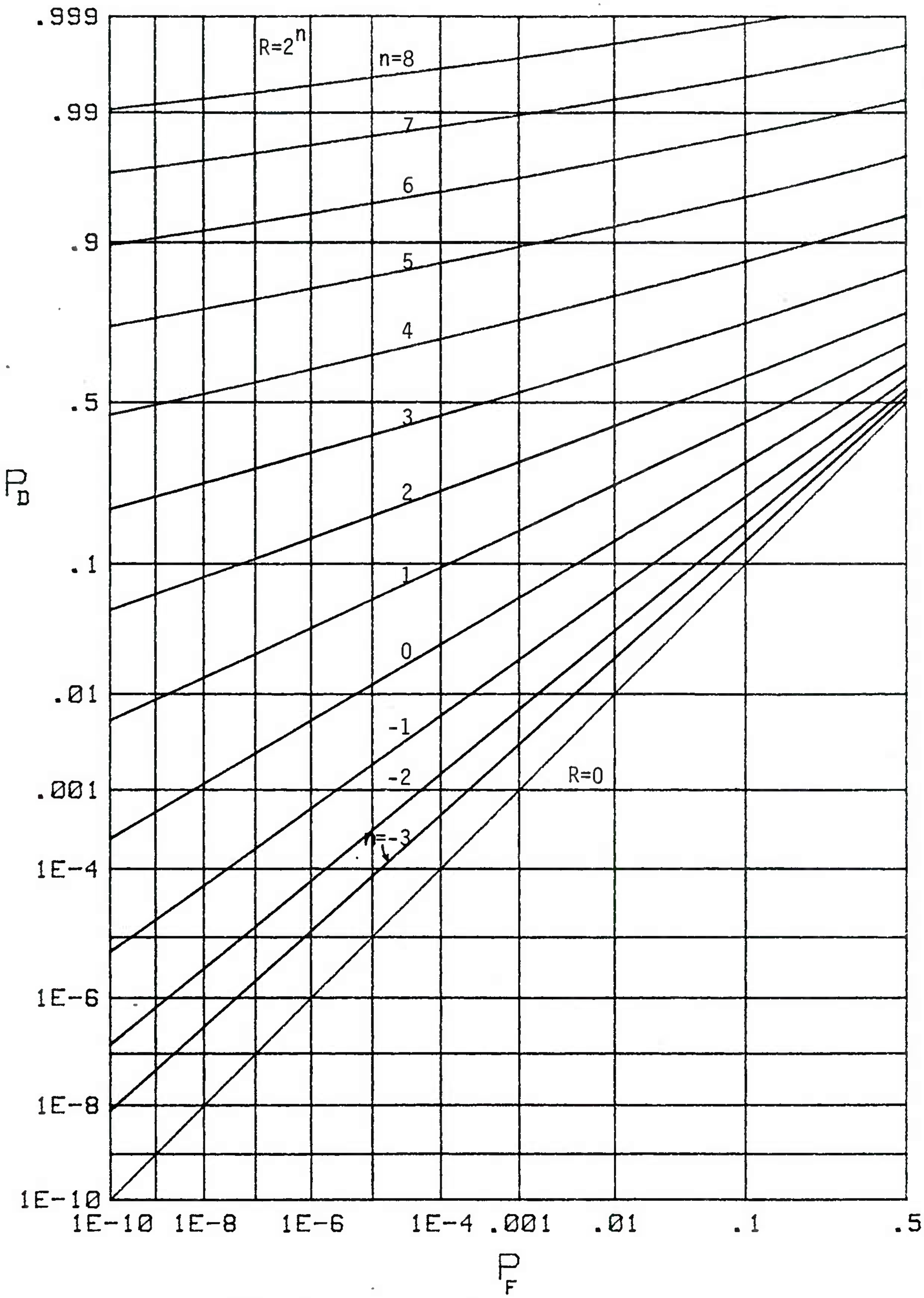

Figure 23. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=4, r=2

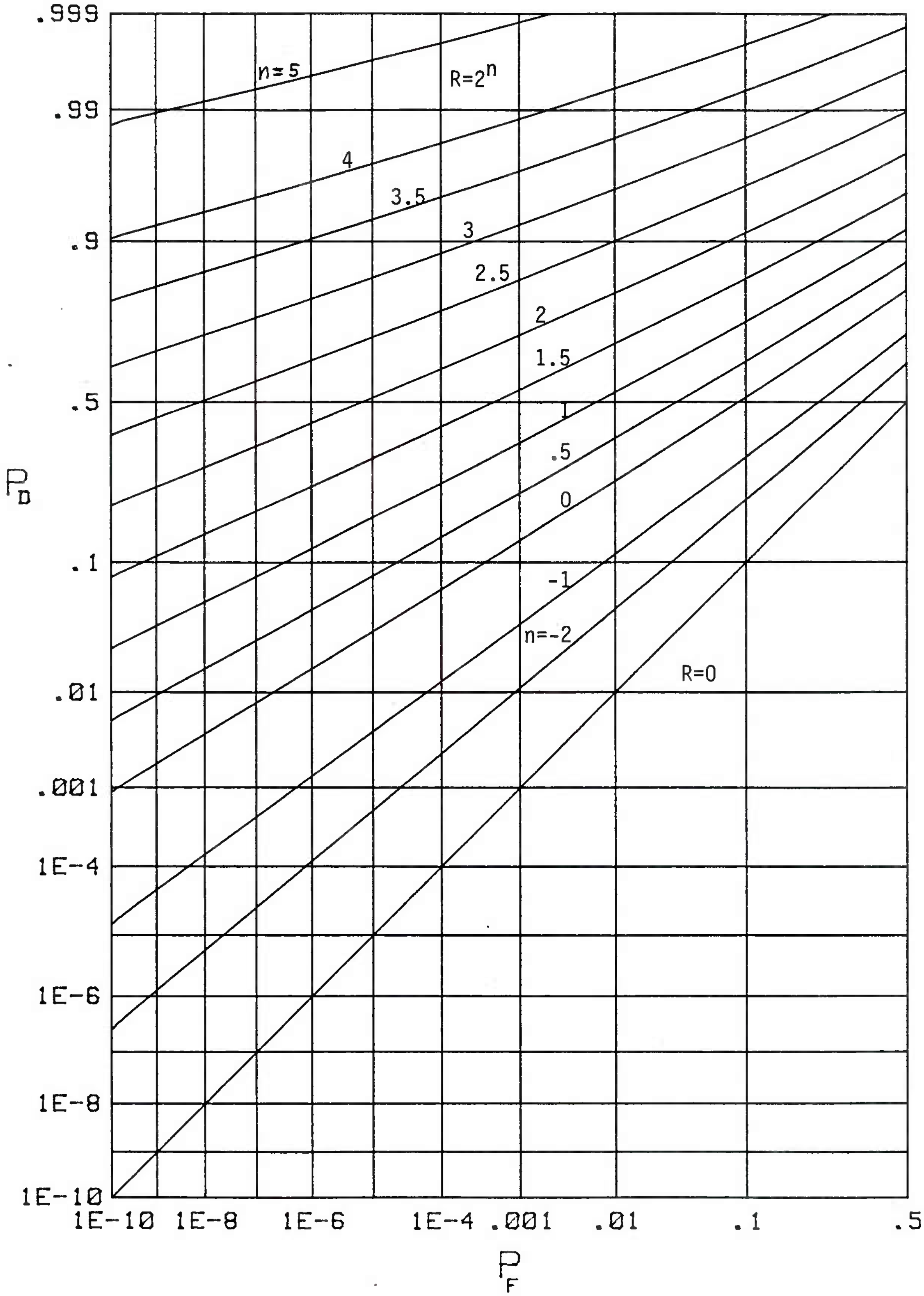

Figure 24. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=8, r=1

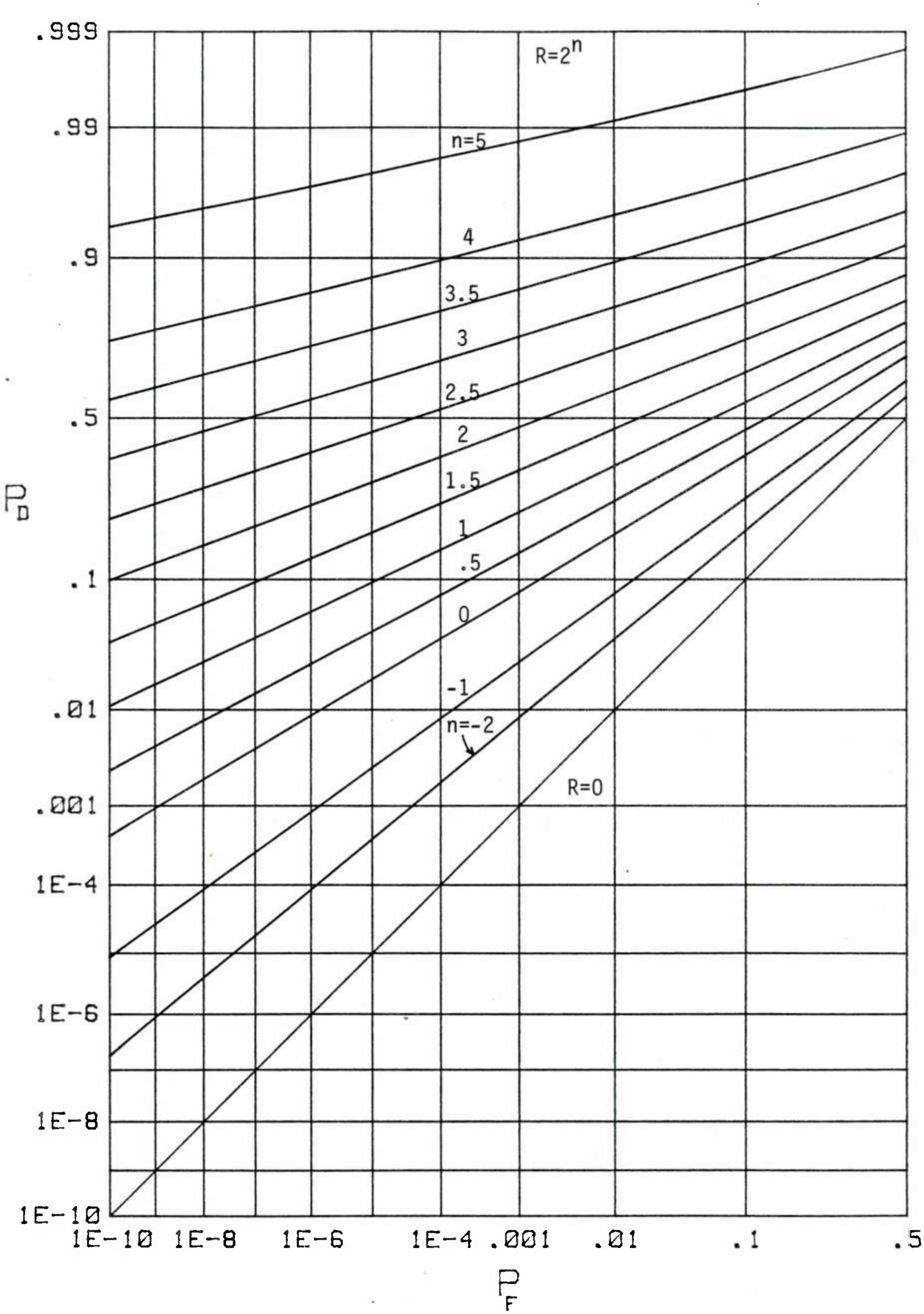

Figure 25. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=8, r=2

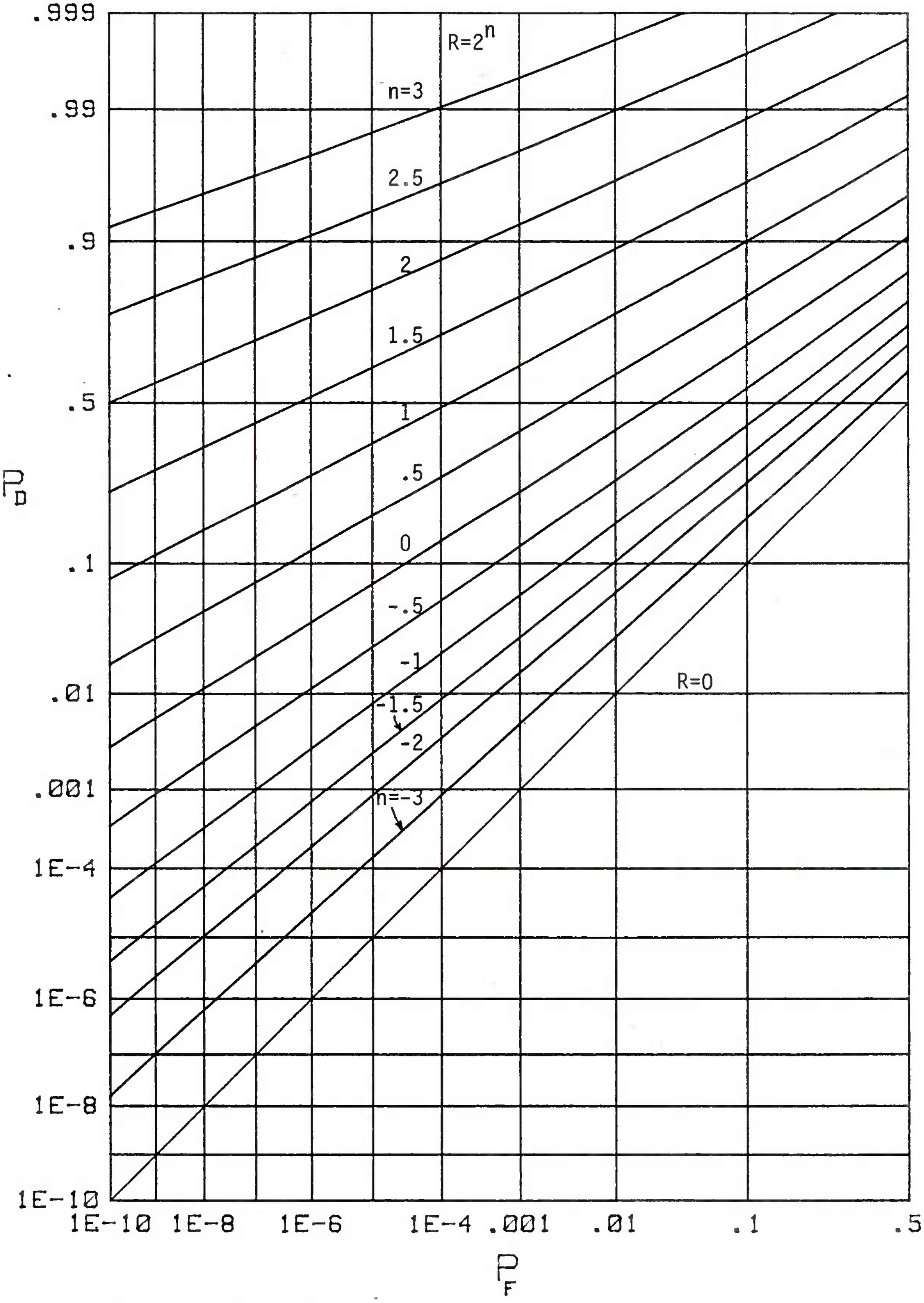

Figure 26. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=16, r=1

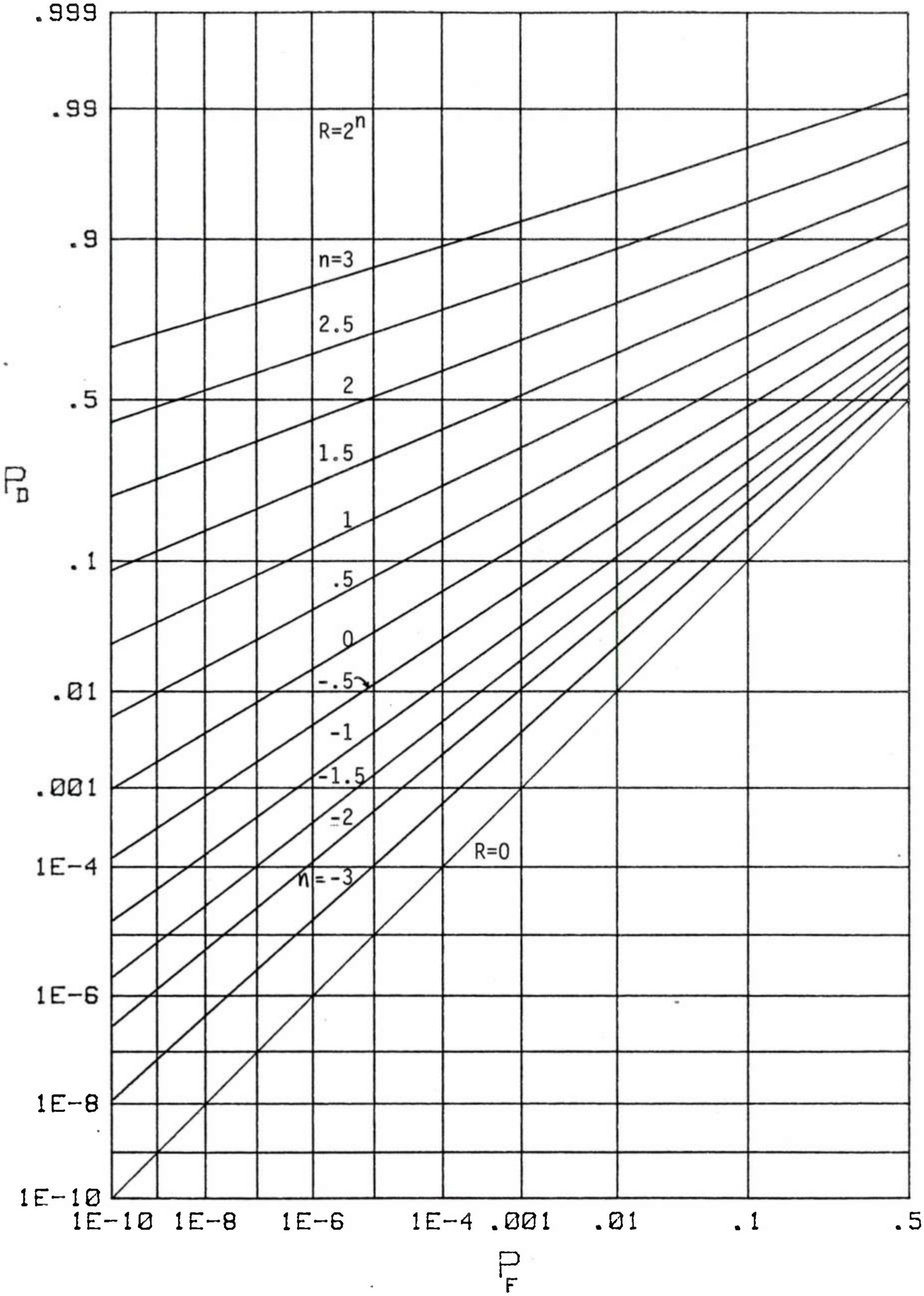

Figure 27. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=16, r=2

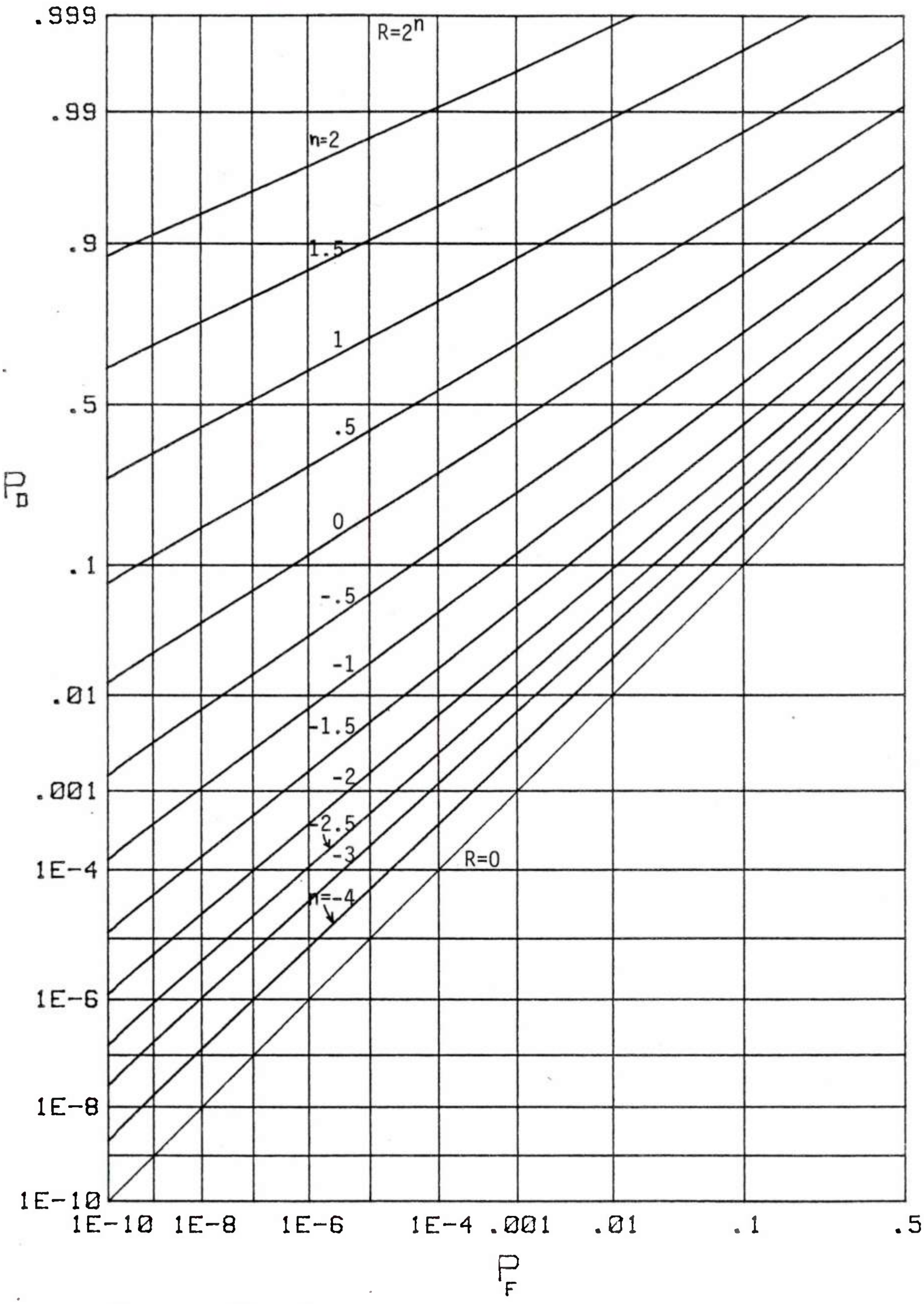

Figure 28. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=32, r=1

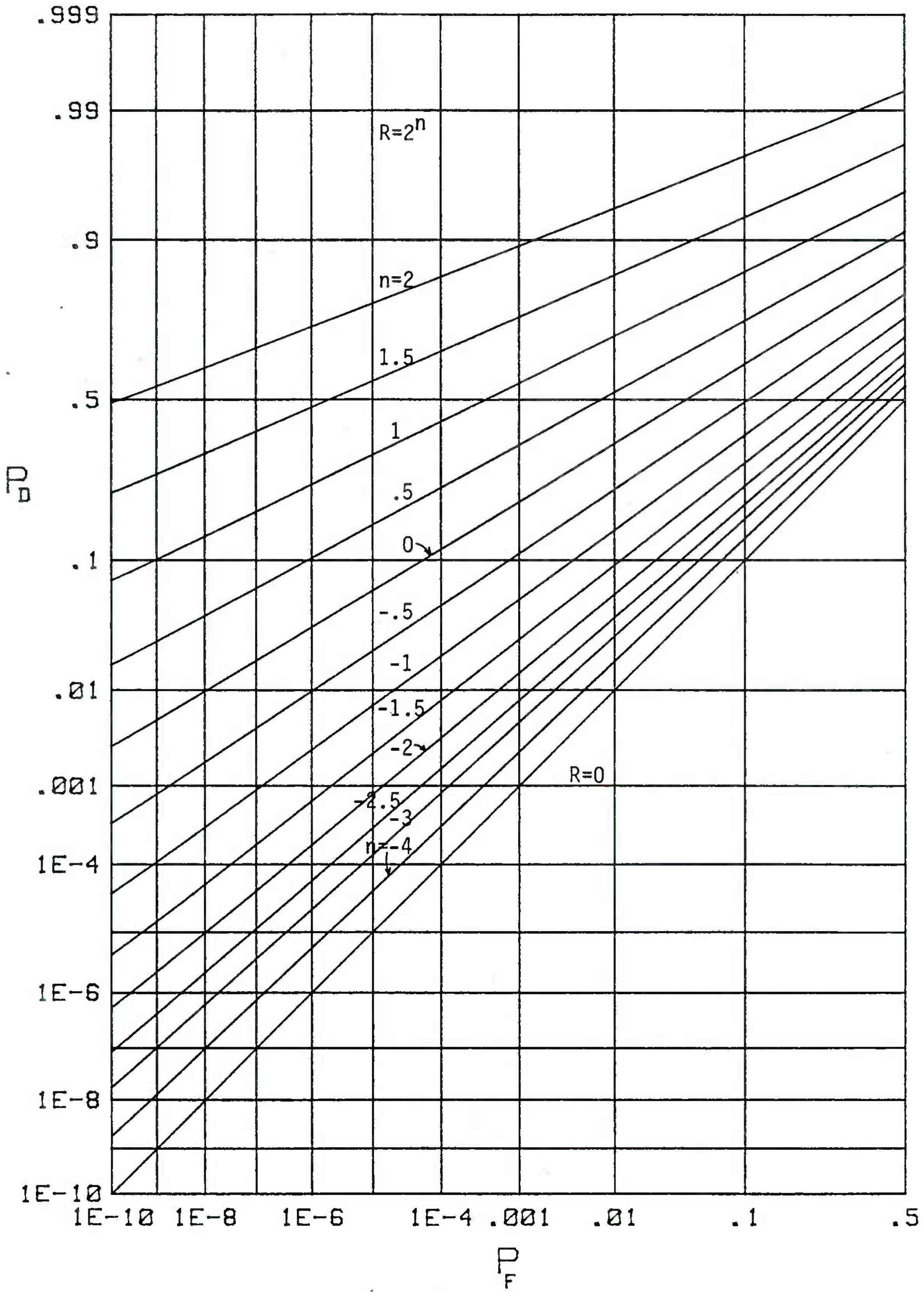

Figure 29. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=32, r=2

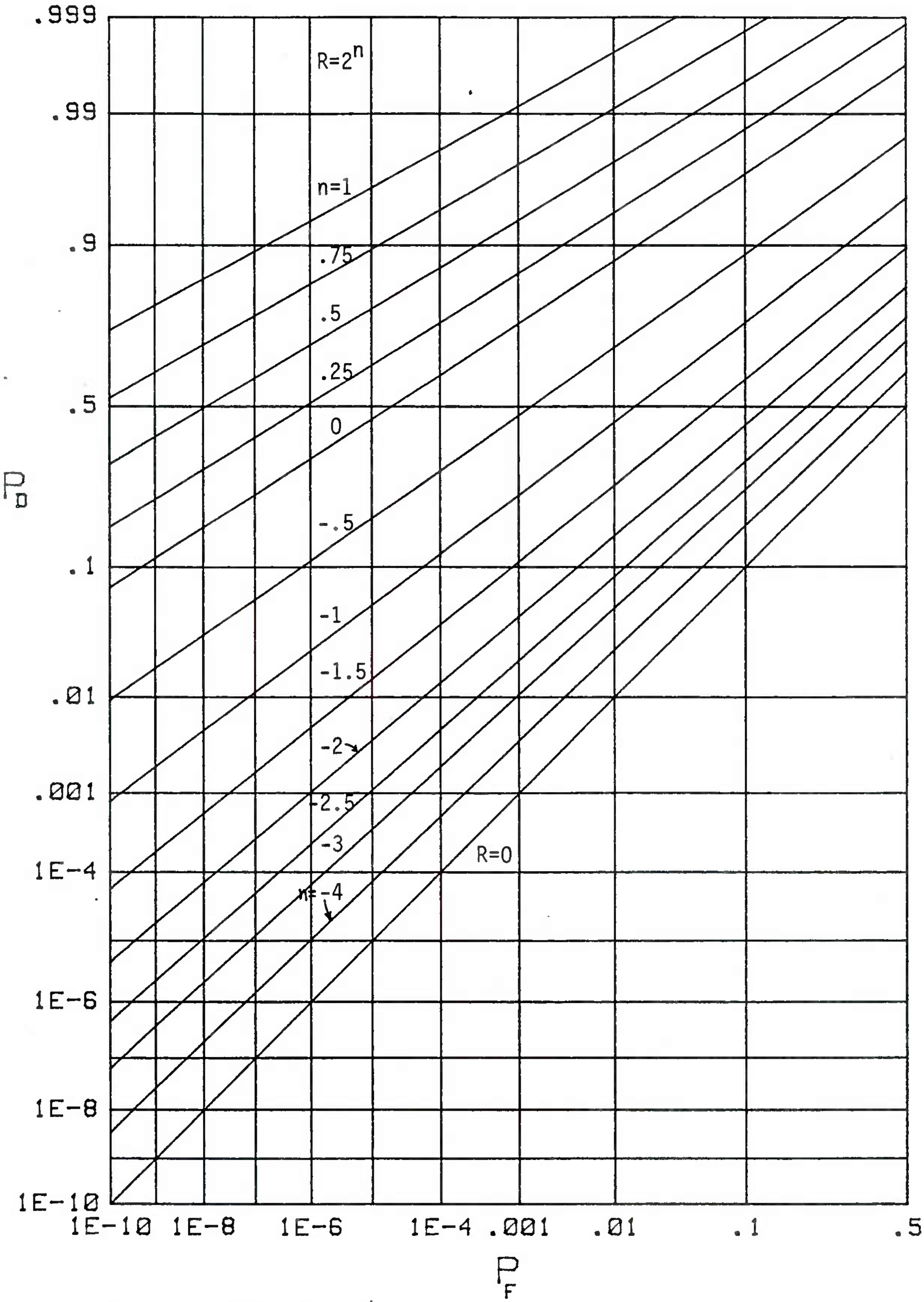

Figure 30. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=64, r=1

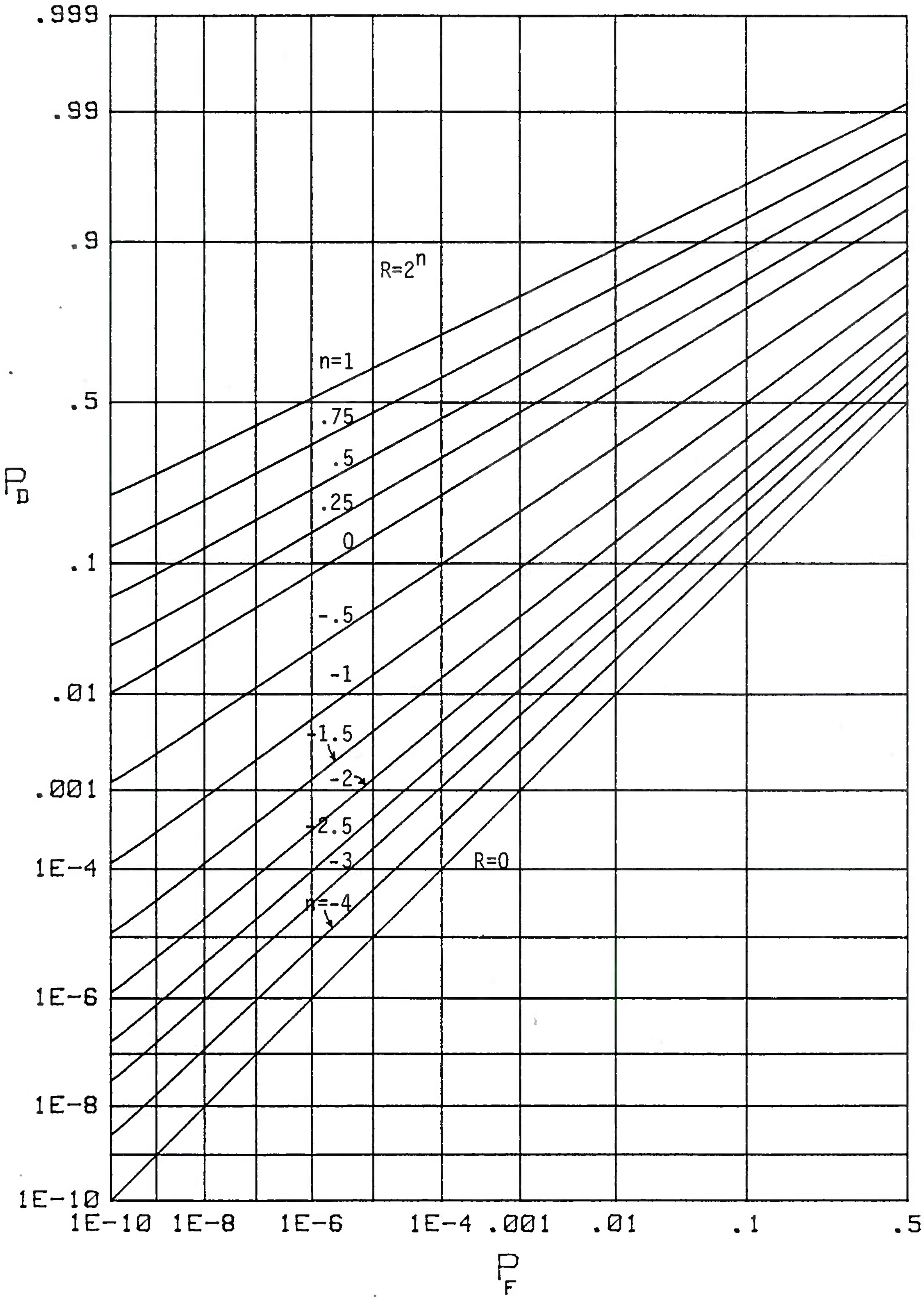

Figure 31. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=64, r=2

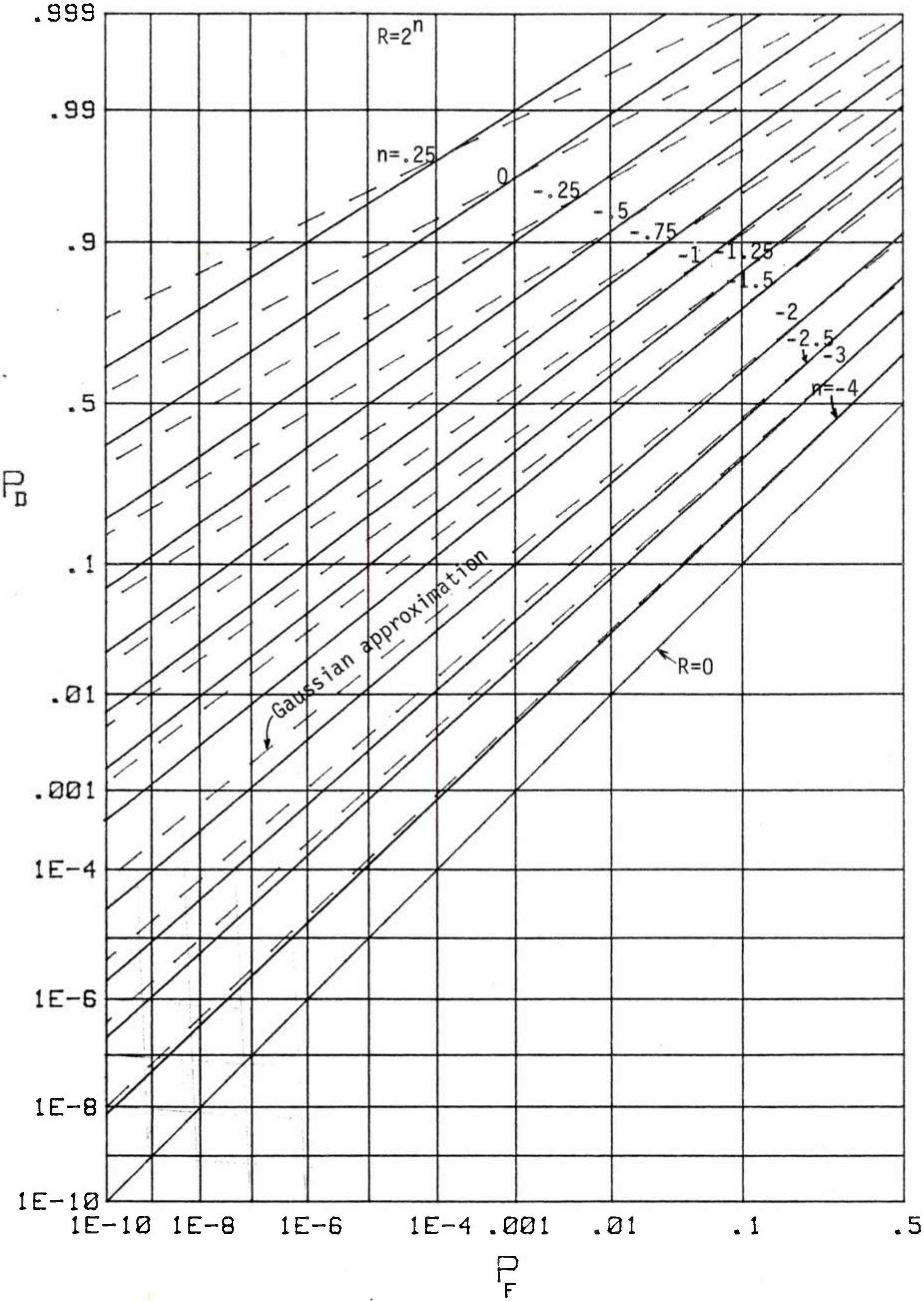

Figure 32. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=128, r=1

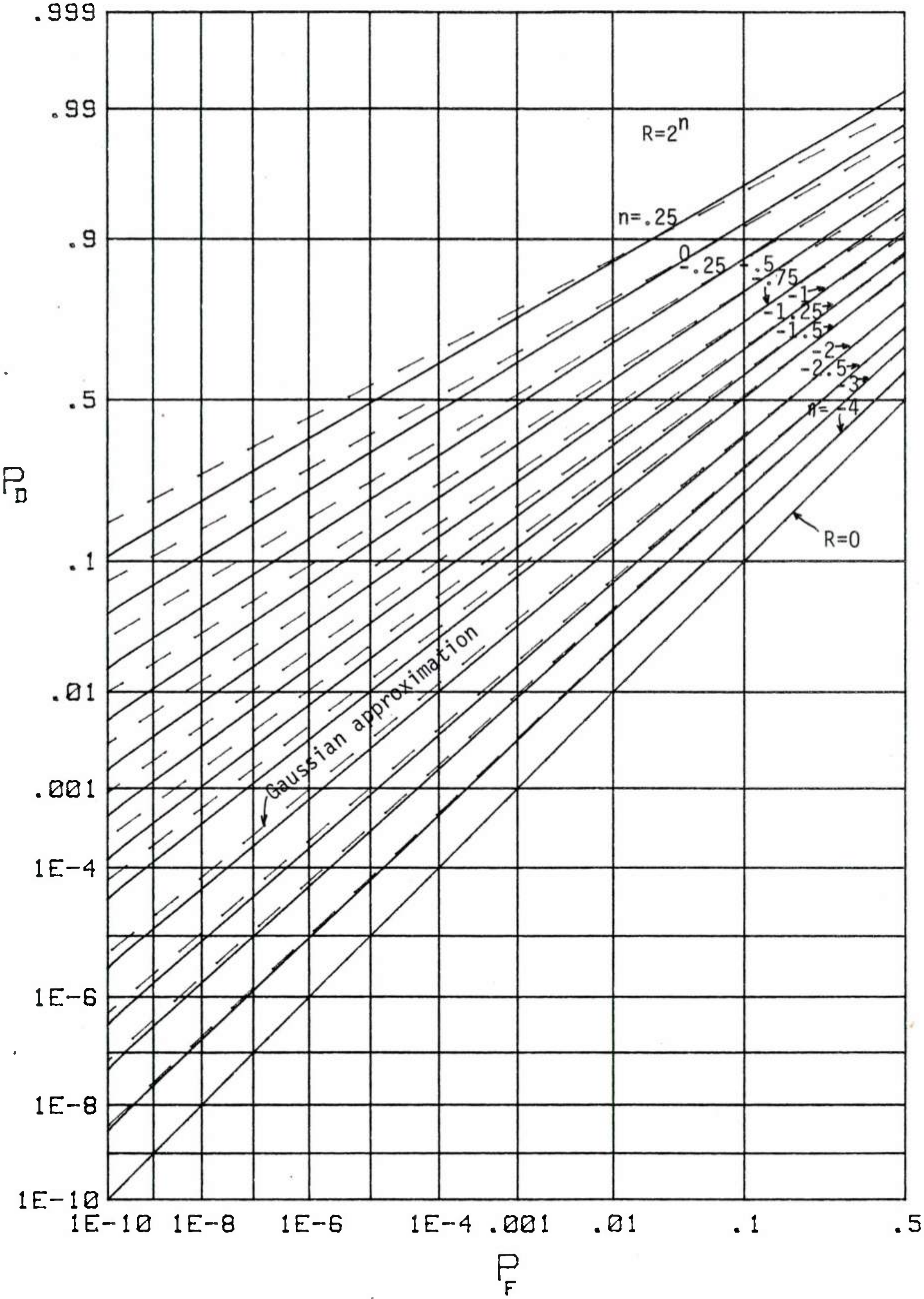

Figure 33. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=128, r=2

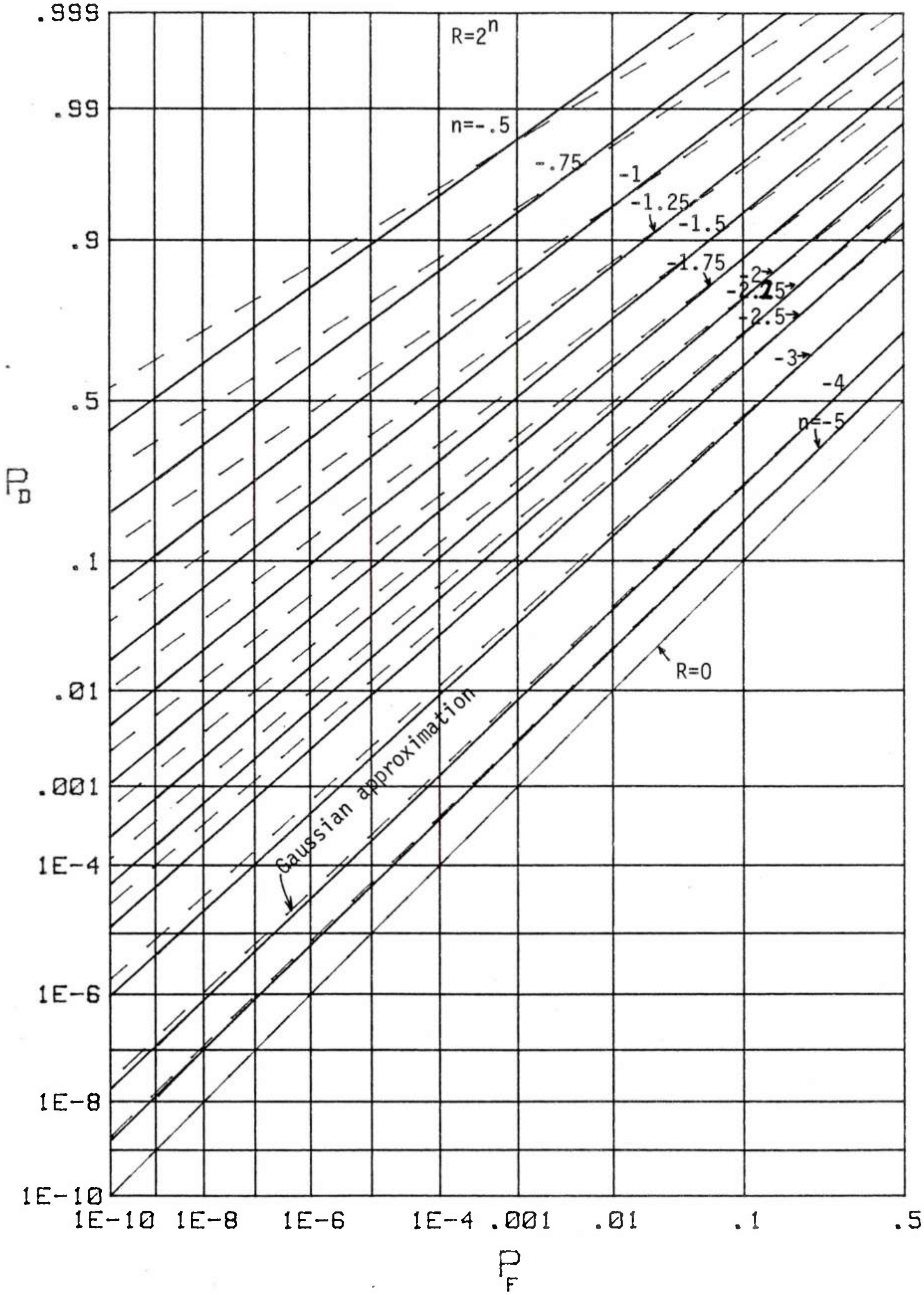

Figure 34. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=256, r=1

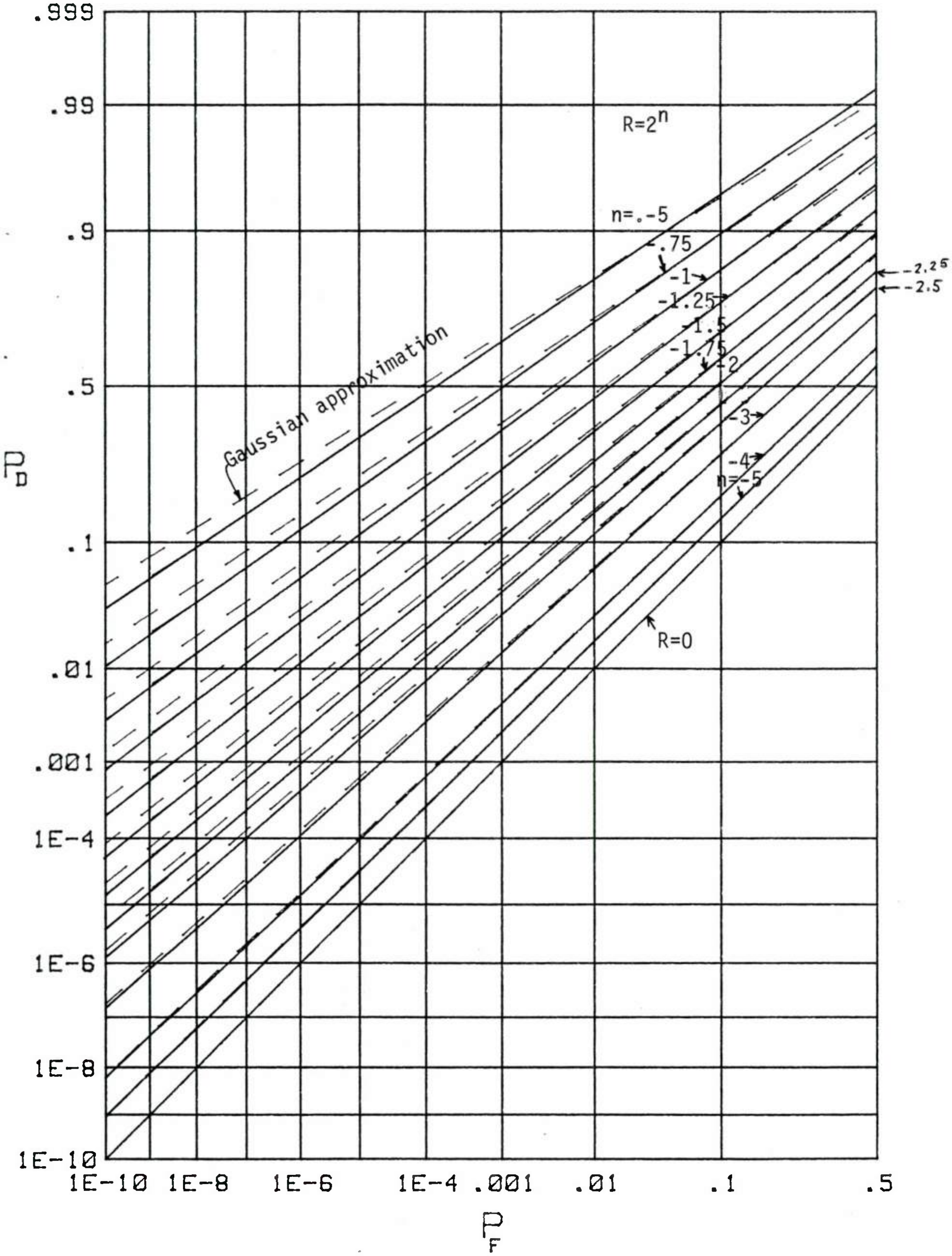

Figure 35. Operating Characteristics for Cross-Correlator without Sample Mean Removal, N=256, r=2

APPENDIX A. CORRELATOR OUTPUT INDEPENDENCE OF MEANS

If we let scale factors $\alpha=1$ but $\beta\neq1$ in (11), we still get $\gamma=1$ from (13). This means that correlator output q in (12) is independent of μ_u, μ_v, β. To see this directly, let

$$u_n = \mu_u + y_n, \quad v_n = \mu_v + z_n , \tag{A-1}$$

where means

$$\overline{y_n} = \overline{z_n} = 0 . \tag{A-2}$$

Then (11) yields, with $\alpha=1$,

$$\tilde{u}_n = y_n - \frac{1}{N}\sum_{m=1}^{N} y_m , \tag{A-3}$$

which is obviously independent of the actual value of mean μ_u. Also, (11) yields

$$\tilde{v}_n = z_n - \frac{\beta}{N}\sum_{m=1}^{N} z_m + \mu_v(1-\beta) , \tag{A-4}$$

which still depends on μ_v and β. When (A-4) is substituted in (12), we get correlator output

$$q = \sum_{n=1}^{N} \tilde{u}_n \left[z_n - \frac{\beta}{N}\sum_{m=1}^{N} z_m + \mu_v(1-\beta) \right] . \tag{A-5}$$

But since

$$\sum_{n=1}^{N} \tilde{u}_n = 0 \tag{A-6}$$

from (A-3), (A-5) reduces to

$$q = \sum_{n=1}^{N} \tilde{u}_n z_n = \sum_{n=1}^{N} \left(y_n - \frac{1}{N} \sum_{m=1}^{N} y_m \right) z_n =$$

$$= \sum_{n=1}^{N} y_n z_n - \frac{1}{N} \sum_{m=1}^{N} y_m \sum_{n=1}^{N} z_n \tag{A-7}$$

in terms of the ac components defined in (A-1) and (A-2). Correlator output (A-7) is obviously independent of means μ_u, μ_v and scale factor β in (11).

APPENDIX B. A USEFUL INTEGRAL OF EXPONENTIALS OF MATRIX FORMS

For symmetric KxK matrix M, with det M>0, the following K-fold integral is well known (see for example, [9, section 8-3]):

$$\int dX \, \exp[-\tfrac{1}{2}X^T M X + N^T X] = \left[\frac{(2\pi)^K}{\det M}\right]^{1/2} \exp\left[\frac{1}{2} N^T M^{-1} N\right] . \quad \text{(B-1)}$$

Here X and N are Kx1 column matrices. We wish to extend this result to the case of double integral

$$I = (2\pi)^{-K} \iint dU \, dV \, \exp\left[-\frac{1}{2} U^T A U - \frac{1}{2} V^T B V + U^T C V + D^T U + E^T V\right] , \quad \text{(B-2)}$$

where A and B are symmetric without loss of generality, and the integral converges; here, matrices A, B, C are KxK while U, V, D, E are Kx1. Notice that if we had the apparently more-general term

$$U^T C_1 V + V^T C_2 U = U^T (C_1 + C_2^T) V , \quad \text{(B-3)}$$

we would simply let $C = C_1 + C_2^T$, and thereby immediately have form (B-2).

To accomplish the evaluation of I in (B-2), identify $M = B$, $N^T = U^T C + E^T$, $X = V$ in (B-1), and thereby evaluate the V-integral in (B-2), with result

$$\int dV \ldots = \left[\frac{(2\pi)^K}{\det B}\right]^{1/2} \exp\left[\frac{1}{2}(U^T C + E^T) B^{-1} (C^T U + E)\right] . \quad \text{(B-4)}$$

Substituting (B-4) in (B-2), regrouping, and using the symmetry of B (and therefore B^{-1}), there follows

$$I = \frac{(2\pi)^{-K/2}}{(\det B)^{1/2}} \int dU \, \exp\left[-\frac{1}{2} U^T (A - C B^{-1} C^T) U + (D^T + E^T B^{-1} C^T) U + \frac{1}{2} E^T B^{-1} E\right] . \quad \text{(B-5)}$$

Now reemploying (B-1) with identifications $M = A - CB^{-1}C^T$, $N = D + CB^{-1}E$, $X = U$, we get a closed form result for (B-5):

$$I = \left[\det(AB - CB^{-1}C^T B)\right]^{-1/2} \exp\left[\frac{1}{2} E^T B^{-1} E + \right.$$

$$\left. + \frac{1}{2}(D+CB^{-1}E)^T (A-CB^{-1}C^T)^{-1} (D+CB^{-1}E)\right]. \qquad \text{(B-6)}$$

This is the desired general result for integral (B-2).

As an aside, there is probably a more symmetric closed form result than (B-6), since if we represent (B-2) by I(A, B, C, D, E), we quickly see, by interchange of dummy variables U and V, that

$$I(A, B, C, D, E) = I(B, A, C^T, E, D). \qquad \text{(B-7)}$$

However, we have not discovered the symmetric form of (B-6). The present form follows as a result of the sequential integration of (B-2), first on V, then on U.

APPENDIX C. PROGRAM FOR CUMULATIVE AND EXCEEDANCE DISTRIBUTION FUNCTIONS VIA CHARACTERISTIC FUNCTION (23)-(24).

The numerical procedure employed in appendices C, D, and G here is heavily based on [3]. The choices of L, Δ, b in lines 90 to 110 to control truncation error and aliasing are also made according to the method of [3]. The parameters in (24) are evaluated once in lines 210-260 so as to minimize computation time. The FFT subroutine used in lines 1030 et seq is listed in [3, pp. B-11 - B-12], and employs a zero-subscripted array. A sample plot of the cumulative and exceedance distribution functions follows the program.

```
10 ! CROSS-CORRELATOR WITH SAMPLE MEAN REMOVAL; NUSC TR 7045
20    N=5                   ! Number of terms summed to yield output
30    Gamma=.5              ! Scale factor in sample mean removal
40    Mu=-.4                ! U channel mean
50    Mv=.3                 ! V channel mean
60    Su=.7                 ! U channel standard deviation
70    Sv=.9                 ! V channel standard deviation
80    Rho=.6                ! Correlation coefficient
90    L=60                  ! Limit on integral of char. function
100   Delta=.12             ! Sampling increment on char. function
110   Bs=.25*(2*PI/Delta)   ! Shift b, as fraction of alias interval
120   Mf=2^8                ! Size of FFT
130   PRINTER IS 0
140   PRINT "L =";L,"Delta =";Delta,"b =";Bs,"Mf =";Mf
150   REDIM X(0:Mf-1),Y(0:Mf-1)
160   DIM X(0:1023),Y(0:1023)
170   Su2=Su*Su                              ! Calculate
180   Sv2=Sv*Sv                              ! parameters
190   T1=1-Gamma
200   T2=T1*T1
210   E1=2*Rho*Su*Sv
220   E2=Su2*Sv2*(1-Rho*Rho)
230   F1=E1*T1
240   F2=E2*T2
250   G1=N*Mu*Mv*T1
260   G2=.5*N*(Su2*Mv*Mv+Sv2*Mu*Mu-E1*Mu*Mv)*T2
270   N1=.5*(N-1)
280   Muq=G1+(N-Gamma)*Rho*Su*Sv             ! Mean of random  variable q
290   Muy=Muq+Bs                             ! Mean of shifted variable y
300   X(0)=0
310   Y(0)=.5*Delta*Muy
320   FOR Ns=1 TO INT(L/Delta)
330   Xi=Delta*Ns                            ! Argument xi of char. function
340   X2=Xi*Xi                               ! Calculation
350   T1=-Xi*F1                              ! of
360   T2=1+X2*F2                             ! characteristic
370   CALL Div(-X2*G2,Xi*G1,T2,T1,A,B)       ! function
380   CALL Log(1+X2*E2,-Xi*E1,C,D)           ! fy(xi)
390   CALL Log(T2,T1,E,F)
400   CALL Exp(A-N1*C-.5*E,B-N1*D-.5*F+Xi*Bs,Fyr,Fyi)
```

```
410   Ms=Ns MOD Mf                      !  Collapsing
420   X(Ms)=X(Ms)+Fyr/Ns
430   Y(Ms)=Y(Ms)+Fyi/Ns
440   NEXT Ns
450   CALL Fft10z(Mf,X(*),Y(*))         !  0 subscript FFT
460   PLOTTER IS "GRAPHICS"
470   GRAPHICS
480   SCALE 0,Mf,-14,0
490   LINE TYPE 3
500   GRID Mf/8,1
510   PENUP
520   LINE TYPE 1
530   B=Bs*Mf*Delta/(2*PI)              !  Origin for random variable q
540   MOVE B,0
550   DRAW B,-14
560   PENUP
570   FOR Ks=0 TO Mf-1
580   T=Y(Ks)/PI-Ks/Mf
590   X(Ks)=.5-T                        !  Cumulative probability in X(*)
600   Y(Ks)=Pr=.5+T                     !  Exceedance probability in Y(*)
610   IF Pr>=1E-12 THEN Y=LGT(Pr)
620   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
630   IF ABS(Pr)<1E-12 THEN Y=-12
640   PLOT Ks,Y
650   NEXT Ks
660   PENUP
670   PRINT Y(0);Y(1);Y(Mf-2);Y(Mf-1)
680   FOR Ks=0 TO Mf-1
690   Pr=X(Ks)
700   IF Pr>=1E-12 THEN Y=LGT(Pr)
710   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
720   IF ABS(Pr)<1E-12 THEN Y=-12
730   PLOT Ks,Y
740   NEXT Ks
750   PENUP
760   PAUSE
770   DUMP GRAPHICS
780   PRINT LIN(5)
790   PRINTER IS 16
800   END
810   !
820   SUB Div(X1,Y1,X2,Y2,A,B)          !  Z1/Z2
830   T=X2*X2+Y2*Y2
840   A=(X1*X2+Y1*Y2)/T
850   B=(Y1*X2-X1*Y2)/T
860   SUBEND
870   !
880   SUB Log(X,Y,A,B)                  !  PRINCIPAL LOG(Z)
890   A=.5*LOG(X*X+Y*Y)
900   IF X<>0 THEN 930
910   B=.5*PI*SGN(Y)
920   GOTO 950
930   B=ATN(Y/X)
940   IF X<0 THEN B=B+PI*(1-2*(Y<0))
950   SUBEND
960   !
```

```
970   SUB Exp(X,Y,A,B)                        !  EXP(Z)
980   T=EXP(X)
990   A=T*COS(Y)
1000  B=T*SIN(Y)
1010  SUBEND
1020  !
1030  SUB Fft10z(N,X(*),Y(*))     ! N <= 2^10 = 1024, N=2^INTEGER     0 SUBSCRIPT
```

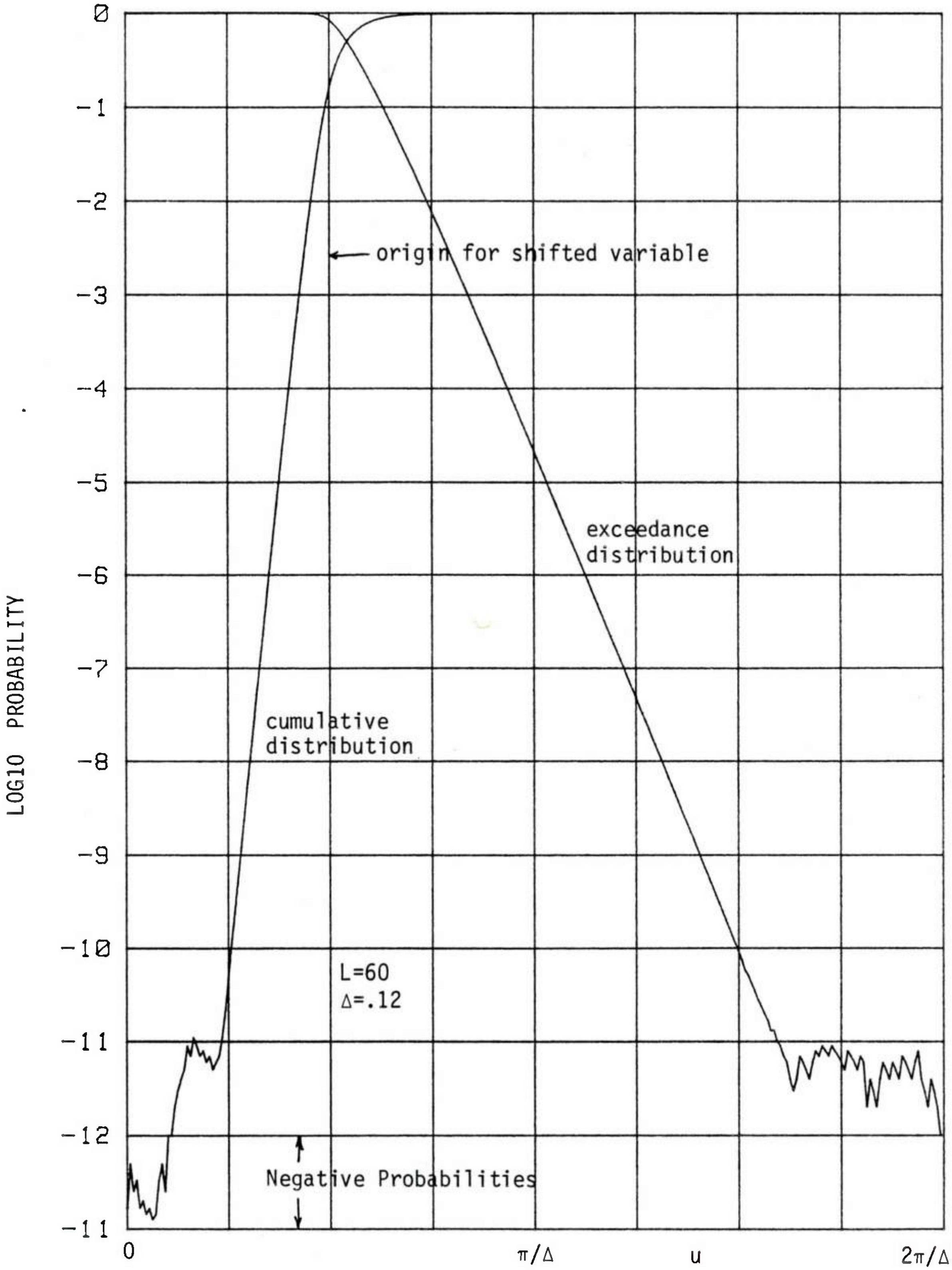

Figure C-1. Cumulative and Exceedance Distribution Functions

APPENDIX D. PROGRAM FOR EVALUATION OF OPERATING CHARACTERISTICS FOR γ=1

The comments in appendix C are relevant here also. The characteristic function used as the starting point is given by (81). Sampling increment Δ employed on the characteristic function can be coarse for small signal-to-noise ratio R, but must be finer for larger R. The quantity Delta0, Δ_0, in line 30 is that used for R=0; all other Δ values are sub-multiples, as indicated by lines 110-130. A table follows.

N	4	6	8	12	16	24	32	48	64	96	128	256
Δ_0	.10	.09	.08	.06	.05	.05	.05	.05	.04	.03	.03	.02

Table D-1. Values of Δ_0 for γ=1

Let Δ_1 denote the sampling increment employed for a particular value of signal-to-noise ratio R_1>0. The cumulative and exceedance distribution function values are available at spacing $s = 2\pi/(M_f\Delta)$ in general, where M_f is the FFT size. If, for example, $\Delta_1 = \Delta_0/2$, then $s_1 = 2s_0$, meaning that probability values occur twice as coarsely for R_1 as for R=0. Then in order to plot detection probability P_D vs false alarm probability P_F without interpolating points, it is necessary to skip every other P_F point available, and only plot those P_F, P_D pairs corresponding to the <u>same</u> threshold. More generally, if $\Delta_1 = \Delta_0/K$, where K is an integer, then $s_1 = Ks_0$, and we plot only every K-th point of the available P_F values. Here we have chosen K to be a power of 2, for the purpose of ease of plotting.

We choose bias (shift) b in line 40 in order to give a random variable y which is virtually always positive for R=0; see [3]. We then keep b fixed as R increases, which makes the probability of y>0 even greater. This feature of choosing the same b for all $R \geq 0$ enables an easy comparison of P_D and P_F, since common threshold values are then conveniently realized.

It was observed under (39) that the characteristic functions (39) or (81) have monotonically decreasing magnitudes for all $\xi \geq 0$. This makes the choice of L, the truncation value on the characteristic function integral in (63) or (64), rather simple; all we need to do is monitor $|f_h(\xi)|$ of (81) until it decreases below a tolerance, here taken as 1E-12. There is no trial-and-error procedure required as in [3] to guarantee negligible truncation error.

Subroutines Exp and Log have already been listed in appendix C, and so are not relisted here.

```
10 !   GAMMA = 1        SAMPLE MEAN REMOVAL
20     Nc=8                         !  N, Number of terms summed
30     Delta0=.08                   !  Initial delta
40     Bs=PI/Delta0                 !  Bias b
50     Mf=2^10                      !  Size of FFT
60     OUTPUT 0;"GAMMA = 1";"     N =";Nc
70     OUTPUT 0;""
80     DATA -2,-1,0,.5,1,1.5,2,2.5,3,3.5,4,5
90     READ Ns(*)                   !  SNR R=2^n
100    OUTPUT 0;Ns(*);
110    DATA 1,2,2,2,4,4,4,8,8,16,16,32,64
120    READ Idelta(*)
130    MAT Delta=(Delta0)/Idelta
140    OUTPUT 0;Delta(*);
150    DATA 1E-10,1E-9,1E-8,1E-7,1E-6,1E-5,1E-4,.001,.01,.1,.5,.9,.99,.999
160    READ Sc(*)
170    DIM Ns(1:12),Idelta(0:12),Delta(0:12),Sc(1:14)
180    DIM X(0:8191),Y(0:8191)
190    FOR I=1 TO 14
200    Sc(I)=FNInvphi(Sc(I))
210    NEXT I
220    S=Sc(1)
230    B=Sc(14)
240    Scale=(B-S)/(0-S)
250    X1=30
260    X2=170
270    Y1=35
280    Y2=Y1+(X2-X1)*Scale
290    PLOTTER IS "9872A"
300    LIMIT X1,X2,Y1,Y2
310    OUTPUT 705;"VS3"
320    SCALE S,0,S,B
330    FOR I=1 TO 14
340    MOVE S,Sc(I)
350    DRAW 0,Sc(I)
360    NEXT I
370    FOR I=1 TO 11
380    MOVE Sc(I),S
390    DRAW Sc(I),B
400    NEXT I
410    MOVE S,S
420    DRAW 0,0
430    PENUP
```

```
440   M1=Mf-1
450   M2=Mf/2
460   FOR In=0 TO 12
470   IF In>0 THEN 500
480   Rc=0
490   GOTO 510
500   Rc=2^Ns(In)                  !  SNR R=2^n
510   OUTPUT 0;"R =";Rc,"    Delta =";Delta(In)
520   ASSIGN #1 TO "ABSCIS"        !  Temporary storage for
530   Delta=Delta(In)              !  false alarm probability
540   R2=Rc*2
550   R21=R2+1
560   N12=(Nc-1)/2
570   Mux=(Nc-1)*Rc                !  Mean of random  variable h
580   Muy=Mux+Bs                   !  Mean of shifted variable y
590   REDIM X(0:M1),Y(0:M1)
600   MAT X=ZER
610   MAT Y=ZER
620   X(0)=0
630   Y(0)=.5*Delta*Muy
640   Ls=0
650   Ls=Ls+1
660   Xi=Delta*Ls                  !  Argument xi of char. fn.
670   CALL Log(1+Xi*Xi*R21,-Xi*R2,Ai,Bi)          !  Calculation
680   CALL Exp(-N12*Ai,Xi*Bs-N12*Bi,Fyr,Fyi)      !  of
690   Ms=Ls MOD Mf                                !  characteristic
700   Ar=Fyr/Ls                                   !  function
710   Ai=Fyi/Ls                                   !  fy(xi)
720   X(Ms)=X(Ms)+Ar
730   Y(Ms)=Y(Ms)+Ai
740   Magsq=Ar*Ar+Ai*Ai
750   IF Magsq>1E-24 THEN 650
760   OUTPUT 0;"Xi =";Xi;"    Mag =";SQR(Magsq)
770   CALL Fft13z(Mf,X(*),Y(*))
780   FOR Ms=0 TO M1
790   T=Y(Ms)/PI-Ms/Mf
800   X(Ms)=.5-T                   !  Cumulative distribution function
810   Y(Ms)=.5+T                   !  Exceedance distribution function
820   NEXT Ms
830   OUTPUT 0;Y(0);Y(1);Y(M1-1);Y(M1)
840   PLOTTER IS "GRAPHICS"
850   GRAPHICS
860   SCALE 0,Mf,-14,0
870   LINE TYPE 3
880   GRID Mf/8,1
890   PENUP
900   LINE TYPE 1
910   FOR Ms=0 TO M1
920   Pr=Y(Ms)
930   IF Pr>=1E-12 THEN Y=LGT(Pr)
940   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
950   IF ABS(Pr)<1E-12 THEN Y=-12
960   PLOT Ms,Y
970   NEXT Ms
980   PENUP
```

```
990   FOR Ms=0 TO M1
1000  Pr=X(Ms)
1010  IF Pr>=1E-12 THEN Y=LGT(Pr)
1020  IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
1030  IF ABS(Pr)<1E-12 THEN Y=-12
1040  PLOT Ms,Y
1050  NEXT Ms
1060  PENUP
1070  DUMP GRAPHICS
1080  OUTPUT 0;""
1090  IF In>0 THEN 1200
1100  FOR Ms=M2 TO M1
1110  IF Y(Ms)<=0 THEN 1130
1120  NEXT Ms
1130  M3=Ms-1
1140  REDIM X(M2:M3)
1150  FOR Ms=M2 TO M3
1160  X(Ms)=FNInvphi(Y(Ms))
1170  NEXT Ms
1180  PRINT #1;X(*)                 !  Store false alarm probability
1190  GOTO 1380
1200  REDIM X(M2:M3)
1210  READ #1;X(*)                  !  Read in false alarm probability
1220  Id=Idelta(In)
1230  J2=M2/Id
1240  J3=INT(M3/Id)+1
1250  FOR J=J2 TO J3
1260  Y(J)=FNInvphi(Y(J))
1270  NEXT J
1280  PLOTTER IS "9872A"
1290  LIMIT X1,X2,Y1,Y2
1300  OUTPUT 705;"VS3"
1310  SCALE S,0,S,B
1320  FOR J=J2 TO J3
1330  T=J*Id
1340  IF T>M3 THEN 1370
1350  PLOT X(T),Y(J)
1360  NEXT J
1370  PENUP
1380  NEXT In
1390  END
1400  !
1410  SUB Exp(X,Y,A,B)                        !  EXP(Z)
1460  !
1470  SUB Log(X,Y,A,B)                        !  PRINCIPAL LOG(Z)
1550  !
1560  DEF FNInvphi(X)        !  INVPHI(X) via AMS 55, 26.2.23
1570  IF (X>0) AND (X<1) THEN 1600
1580  P=9.99999999999E90*(2*X-1)
1590  GOTO 1670
1600  IF X=.5 THEN RETURN 0
1610  P=X
1620  IF X>.5 THEN P=.5-(X-.5)
1630  P=SQR(-2*LOG(P))
1640  T=1+P*(1.432788+P*(.189269+P*.001308))
1650  P=P-(2.515517+P*(.802853+P*.010328))/T
1660  IF X<.5 THEN P=-P
1670  RETURN P
1680  FNEND
1690  !
1700  SUB Fft13z(N,X(*),Y(*))    ! N <= 2^13, N=2^INTEGER, 0 SUBSCRIPT
```

APPENDIX E. ASYMPTOTIC EXPANSIONS FOR DISTRIBUTIONS WHEN r>0

The characteristic function of interest is given by (100) and (102):

$$f_h(\xi) = (1+i\xi)^{-\nu} (1-i\xi\omega)^{-\nu} \exp\left[\frac{i\xi\eta}{1-i\xi\omega}\right], \tag{E-1}$$

where for notational convenience in this appendix, we let

$$\nu = \frac{N}{2}, \quad \eta = Nr^2, \quad \omega = 1+2R \ . \tag{E-2}$$

The cumulative distribution function is obtained by substitution of (E-1) in (63):

$$P_h(u) = \frac{-1}{i2\pi} \int_{C_+} d\xi \ \xi^{-1}(1+i\xi)^{-\nu} (1-i\xi\omega)^{-\nu} \exp\left[\frac{i\xi\eta}{1-i\xi\omega} - iu\xi\right] . \tag{E-3}$$

The ν-th powers are principal value, being positive real where C_+ crosses the positive imaginary axis.

Now let $z = 1+i\xi$ in (E-3), yielding

$$P_h(u) = \frac{-1}{i2\pi} \int_{C_1} dz \ (z-1)^{-1} z^{-\nu} (1+\omega-\omega z)^{-\nu} \exp\left[\frac{(z-1)\eta}{1+\omega-\omega z} - u(z-1)\right] . \tag{E-4}$$

The contours C_+ and C_1 in (E-3) and (E-4) are depicted as dashed lines in figure E-1. The pole at $\xi=0$ is moved to $z=1$; the remaining singularities are branch points (ν non-integer); the ν-th powers are positive real where C_1 crosses the positive real axis. For $u<0$, an equivalent contour to C_1 is that indicated by C_2 in figure E-1, since the exponential in (E-4) furnishes rapid decay in the left-half z-plane. We write (E-4) in the form

$$P_h(u) = \frac{\exp(u)}{i2\pi} \int_{C_2} dz \ z^{-\nu} \exp(-uz) \ g_1(z) \ , \tag{E-5}$$

where

$$g_1(z) = (1-z)^{-1}\ (1+\omega-\omega z)^{-\nu}\ \exp\left[\frac{(z-1)\eta}{1+\omega-\omega z}\right]. \tag{E-6}$$

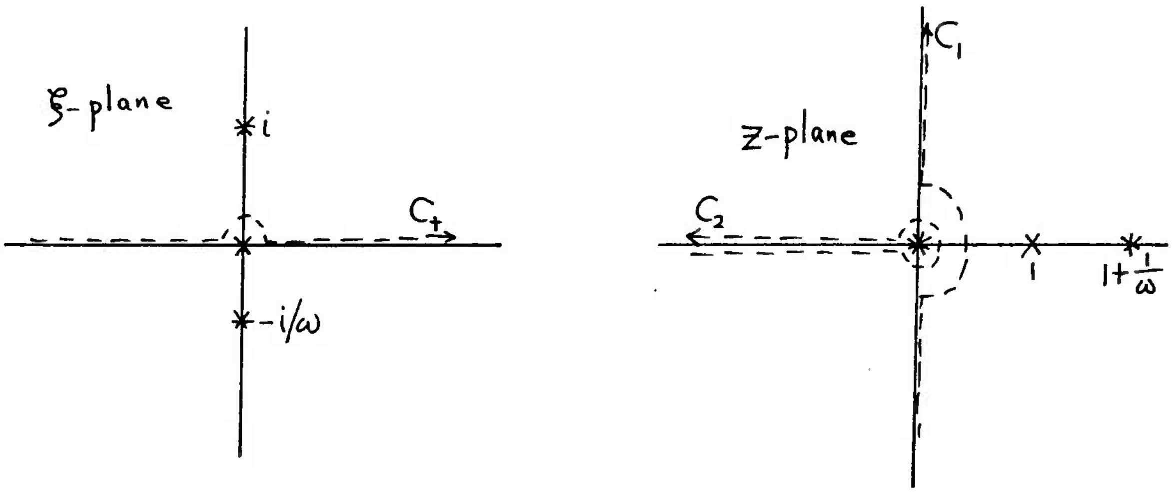

Figure E-1. Contours of Integration for Cumulative Distribution Function

In order to get the asymptotic development of (E-5), we expand $g_1(z)$ in a power series in z,

$$g_1(z) = g_1(0) + g_1'(0)\ z + \dots , \tag{E-7}$$

where

$$g_1(0) = (1+\omega)^{-\nu}\ \exp\left(\frac{-\eta}{1+\omega}\right),\ \frac{g_1'(0)}{g_1(0)} = 1 + \frac{\nu\omega}{1+\omega} + \frac{\eta}{(1+\omega)^2}\ . \tag{E-8}$$

Appeal to [10, p. 96, (4)] then yields (for all ν)

$$P_h(u) \sim \exp(u)\left[\frac{(-u)^{\nu-1}}{\Gamma(\nu)}\ g_1(0) + \frac{(-u)^{\nu-2}}{\Gamma(\nu-1)}\ g_1'(0)\right] =$$

$$= \Gamma(\nu)^{-1}\ (1+\omega)^{-\nu}\ (-u)^{\nu-1}\ \exp\left[u - \frac{\eta}{1+\omega}\right]\left[1 - \frac{\nu-1}{u}\left(1 + \frac{\nu\omega}{1+\omega} + \frac{\eta}{(1+\omega)^2}\right)\right]$$

$$\text{as } u \rightarrow -\infty\ . \tag{E-9}$$

Substitution of (E-2) in (E-9) then yields result (104).

When (E-1) is substituted into (64) instead, we obtain the exceedance distribution function in the form

$$1 - P_h(u) = \frac{1}{i2\pi} \int_{C_-} d\xi\, \xi^{-1} (1+i\xi)^{-\nu} (1-i\xi\omega)^{-\nu} \exp\left[\frac{i\xi n}{1-i\xi\omega} - iu\xi\right] . \qquad \text{(E-10)}$$

Now let $z = \frac{1}{\omega} - i\xi$, to get

$$1 - P_h(u) = \frac{-1}{i2\pi} \int_{C_3} dz \left(z - \frac{1}{\omega}\right)^{-1} \left(1 + \frac{1}{\omega} - z\right)^{-\nu} (\omega z)^{-\nu} *$$

$$* \exp\left[\frac{(1-\omega z)n}{\omega^2 z} + u\left(z - \frac{1}{\omega}\right)\right] . \qquad \text{(E-11)}$$

The contours C_- and C_3 in (E-10) and (E-11) are depicted as dashed lines in figure E-2. The pole at $\xi=0$ is moved to $z = 1/\omega$; the remaining singularities are branch points.

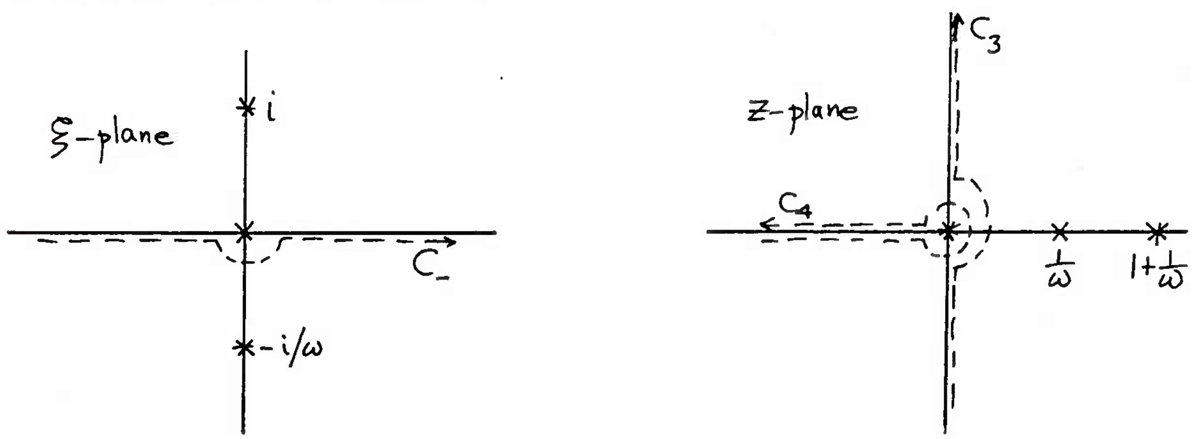

Figure E-2. Contours of Integration for Exceedance Distribution Function

For u>0, an equivalent contour to C_3 is that indicated by C_4 in figure E-2, since the exponential in (E-11) furnishes rapid decay in the left-half z-plane. We write (E-11) in the form

$$1-P_h(u) = \omega\ (1+\omega)^{-\nu} \exp\left(-\frac{u+\eta}{\omega}\right) \frac{1}{i2\pi} \int_{C_4} dz\ z^{-\nu} \exp\left[uz + \frac{\eta}{\omega^2 z}\right] g_2(z)\ , \quad \text{(E-12)}$$

where

$$g_2(z) = (1-\omega z)^{-1} \left(1 - \frac{\omega z}{1+\omega}\right)^{-\nu} . \quad \text{(E-13)}$$

As above, we expand $g_2(z)$ in a power series in z,

$$g_2(z) = 1 + \omega(1 + \frac{\nu}{1+\omega})\ z + \ldots \quad \text{(E-14)}$$

and substitute it in (E-12); employment of [10, p. 105, (2)] now yields (for all ν)

$$1 - P_h(u) \sim \omega\ (1+\omega)^{-\nu} \exp\left(-\frac{u+\eta}{\omega}\right) *$$

$$* \left[\left(\omega\sqrt{\frac{u}{\eta}}\right)^{\nu-1} I_{\nu-1}\left(\frac{2\sqrt{\eta u}}{\omega}\right) + \omega\ (1 + \frac{\nu}{1+\omega}) \left(\omega\sqrt{\frac{u}{\eta}}\right)^{\nu-2} I_{\nu-2}\left(\frac{2\sqrt{\eta u}}{\omega}\right)\right]$$
$$\text{as } u \rightarrow +\infty, \quad \text{(E-15)}$$

where $I_\mu(z)$ is the modified Bessel function of the first kind. This is the general result for the exceedance distribution function; the various parameters given in (E-2) relate it back to the problem of interest in the main text.

As a check on this result, we let $r \rightarrow 0$; then $\eta \rightarrow 0$, and (E-15) reduces, via [6, 9.6.7], to

$$1 - P_h(u) \sim \Gamma(\nu)^{-1}\ \omega\ (1+\omega)^{-\nu}\ u^{\nu-1} \exp(-u/\omega) \left[1 + \frac{\nu-1}{u}\ \omega \left(1 + \frac{\nu}{1+\omega}\right)\right]$$
$$\text{as } u \rightarrow +\infty;\ r=0. \quad \text{(E-16)}$$

Employing the identifications in (E-2), there follows from (E-16)

$$1 - P_h(u) \sim \frac{1+2R}{2(1+R)\Gamma(N/2)} \left(\frac{u}{2(1+R)}\right)^{\frac{N-2}{2}} \exp\left(\frac{-u}{1+2R}\right) *$$

$$* \left[1 + \frac{1+2R}{1+R} \frac{(N-2)(N+4+4R)}{8u}\right] \quad \text{as } u \to +\infty \text{; } r=0 \text{ .} \tag{E-17}$$

In order to compare this result with that for $\gamma=1$, sample mean removal, we must replace N here by N-1; see the paragraph under (101). When this is done, (E-17) reverts precisely to (61) and (62).

Returning to the general result for the exceedance distribution function in (E-15), if we keep $\eta>0$ and use [6, 9.7.1] for large arguments of $I_\mu(z)$, there follows the simpler (less accurate) result

$$1 - P_h(u) \sim \left[2\pi^{1/2}(1+\omega)^{\nu} \, \eta^{\frac{\nu}{2} - \frac{1}{4}}\right]^{-1} \omega^{\nu+\frac{1}{2}} *$$

$$* \, u^{\frac{\nu}{2} - \frac{3}{4}} \exp\left[-\frac{1}{\omega}\left(u^{1/2} - \eta^{1/2}\right)^2\right] \quad \text{as } u \to +\infty. \tag{E-18}$$

When (E-2) is substituted in (E-18), the result quoted in (105) follows.

As a special case of (E-18), if $\nu=1$ (i.e. N=2), then

$$1 - P_h(u) \sim \frac{\omega^{3/2}}{2\pi^{1/2}(1+\omega)(\eta u)^{1/4}} \exp\left[-\frac{1}{\omega}\left(u^{1/2} - \eta^{1/2}\right)^2\right] \text{ as } u \to +\infty \text{ ; } \nu=1. \tag{E-19}$$

APPENDIX F. EXCEEDANCE DISTRIBUTION FUNCTION FOR $\gamma=0$, $N=1$, $r>0$

CHARACTERISTIC FUNCTION APPROACH

When characteristic function (100) with N=1 is substituted in (64), the expression for the exceedance distribution function becomes

$$1 - P_h(u) = \frac{1}{i2\pi} \int_{C_-} d\xi \; \xi^{-1} \; (1+i\xi)^{-1/2} \; (1-i\xi\omega)^{-1/2} \exp\left[\frac{i\xi r^2}{1-i\xi\omega} - iu\xi\right], \quad \text{(F-1)}$$

where $\omega = 1+2R$ as in (102). The square roots are taken as +1 at $\xi=0$. For $u\geq0$, the contour C_- can be modified to that indicated in figure F-1, where the contributions of the large circular arcs in the lower-half ξ-plane tend to zero. The small circle of radius ρ centered at branch point $\xi = -i/\omega$ must have $0 < \rho < 1/\omega$, since the latter is the distance to the pole at the origin.

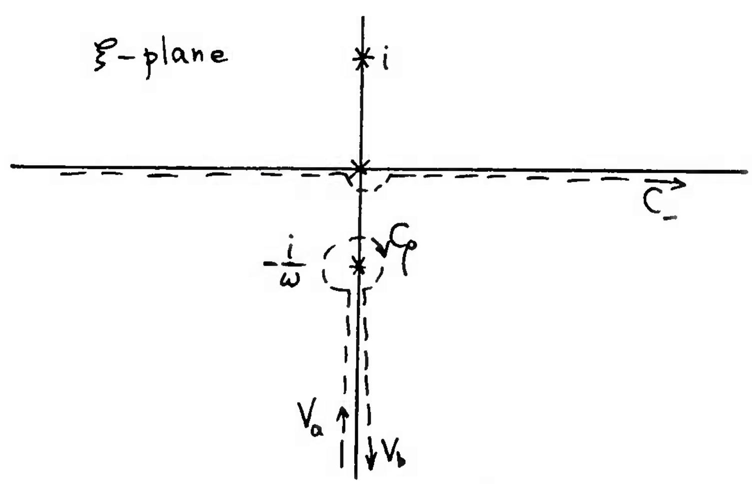

Figure F-1. Equivalent Contours for (F-1)

It is easy to show that the two vertical contributions in figure F-1 are equal. Under the change of variable

$$i\xi = \frac{1+t^2}{\omega}, \quad \text{(F-2)}$$

the sum of the two vertical contributions to exceedance distribution function (F-1) becomes

$$V(\rho) = \frac{2\omega^{1/2}}{\pi} \exp\left(-\frac{u+r^2}{\omega}\right) \int_{(\omega\rho)^{1/2}}^{+\infty} dt\ (1+t^2)^{-1} \ *$$

$$* \ (1+\omega+t^2)^{-1/2} \exp\left(-\frac{u}{\omega}t^2 - \frac{r^2}{\omega t^2}\right). \tag{F-3}$$

This integral remains convergent even as $\rho \to 0+$. Furthermore, the integrand decays rapidly, has no cusps, and involves only elementary functions; also the integral is a sum of positive quantities and retains significance even for large u.

On the small circular contour C_ρ in figure F-1, let

$$i\xi = \frac{1}{\omega} - \rho \exp(i\theta), \tag{F-4}$$

to obtain, for the circular contribution to the exceedance distribution function, the quantity

$$C(\rho) = \frac{\omega}{2\pi(1+\omega)^{1/2}} \exp\left(-\frac{u+r^2}{\omega}\right) \rho^{1/2} \int_{-\pi}^{\pi} d\theta\ (1-\omega\rho E)^{-1} \ *$$

$$*\left(1 - \frac{\omega\rho}{1+\omega} E\right)^{-1/2} \exp\left(i\frac{\theta}{2} + \frac{\lambda}{\rho}E^* + \rho u E\right), \tag{F-5}$$

where we define in this appendix

$$\lambda = \frac{r^2}{\omega^2} = \left(\frac{r}{1+2R}\right)^2, \quad E = \exp(i\theta) \ . \tag{F-6}$$

The exceedance distribution function is given exactly by the sum of (F-3) and (F-5), for any $0 < \rho < 1/\omega$. It would be advantageous numerically to let $\rho \rightarrow 0+$ in these two equations; however, the limit of (F-5) is not obvious and can easily be done incorrectly.

AN ERRONEOUS APPROACH FOR $C(\rho)$

It is tempting to let $\rho \rightarrow 0+$ in those locations in (F-5) where it will "do no damage", obtaining for the integral with scale factor $\rho^{1/2}$ the quantity

$$I_\rho \equiv \rho^{1/2} \int_{-\pi}^{\pi} d\theta \, \exp\left[i \frac{\theta}{2} + \frac{\lambda}{\rho} \exp(-i\theta)\right] . \qquad \text{(F-7)}$$

(The fallacy of doing this for a residue calculation with an essential singularity is demonstrated in the next subsection.) Furthermore, the limit of (F-7) as $\rho \rightarrow 0+$ can in fact be determined in closed form, as follows. Observe that the integrand of (F-7) has a saddle point in the complex θ-plane at

$$\theta_s = -iL \; , \quad L = \ln\left(\frac{2\lambda}{\rho}\right) \; ; \qquad \text{(F-8)}$$

this is in fact the only saddle point in the $(-\pi,\pi)$ strip in the θ-plane. Now let $z = \theta - \theta_s$, getting for (F-7)

$$I_\rho = (2\lambda)^{1/2} \int_{-\pi+iL}^{\pi+iL} dz \, \exp\left[i \frac{z}{2} + \frac{1}{2} \exp(-iz)\right] . \qquad \text{(F-9)}$$

The radius ρ now appears only in the limit L of the integral, and the integrand has a saddle point at z=0.

The straight line contour for (F-9) can be deformed into contour C, depicted in figure F-2, which goes through the saddle point at z=0. Now if

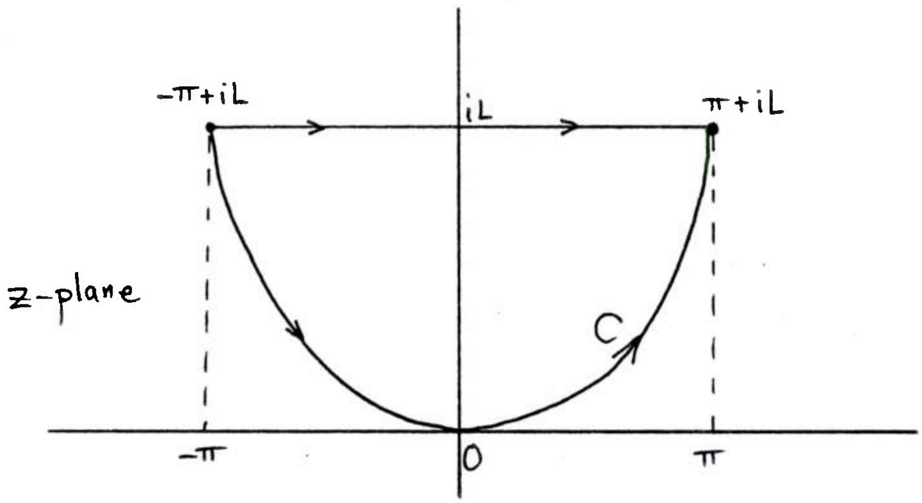

Figure F-2. Equivalent Contours for (F-9)

$\rho \to 0+$, then $L \to +\infty$, and (F-9) yields

$$I_0 = (2\lambda)^{1/2} \int_{-\pi+i\infty}^{\pi+i\infty} dz \, \exp\left[i\,\frac{z}{2} + \frac{1}{2}\exp(-iz)\right] , \tag{F-10}$$

where the contour is the limit of C in figure F-2 as $L \to +\infty$; that is, the integral is between the two valleys at $\pm\pi+i\infty$ and is connected by the saddle point at z=0.

The steepest descent curves out of the saddle point of the integrand of (F-10) are given explicitly by

$$y = -\ln\left(\frac{\sin x}{x}\right) \quad \text{for } -\pi < x < \pi \,. \tag{F-11}$$

Thus if we let

$$z = x + iy = x - i\,\ell n\left(\frac{\sin x}{x}\right) ,$$

$$\frac{dz}{dx} = 1 - i\left(\frac{\cos x}{\sin x} - \frac{1}{x}\right) , \tag{F-12}$$

on the steepest descent curves in (F-10), there follows

$$I_o = (2\lambda)^{1/2} \int_{-\pi}^{\pi} dx\left(1 - i\,\frac{\cos x}{\sin x} + i\,\frac{1}{x}\right) \exp\left[i\,\frac{x}{2} + \frac{1}{2}\,\ell n\left(\frac{\sin x}{x}\right) + \frac{1}{2}\,\frac{x}{\sin x}\,\exp(-ix)\right] =$$

$$= (2\lambda)^{1/2} \int_{-\pi}^{\pi} dx\left(1 - i\,\frac{\cos x}{\sin x} + i\,\frac{1}{x}\right)\left(\frac{\sin x}{x}\right)^{1/2} \exp\left[\frac{x\,\cos x}{2\,\sin x}\right] =$$

$$= (2\lambda)^{1/2}\, 2\int_{0}^{\pi} dx\,\left(\frac{\sin x}{x}\right)^{1/2} \exp\left(\frac{x\,\cos x}{2\,\sin x}\right) = (2\lambda)^{1/2}\, 2(2\pi)^{1/2} = 4(\lambda\pi)^{1/2} . \tag{F-13}$$

(The integral value of $(2\pi)^{1/2}$ in (F-13) was deduced by numerical integration.)

Recalling the definition of I_ρ in (F-7), we then have the dubious result for the limit of (F-5):

$$C(0) \overset{?}{=} \left(\frac{2}{\pi}\right)^{1/2} \frac{r}{(1+R)^{1/2}} \exp\left(-\,\frac{u+r^2}{1+2R}\right), \tag{F-14}$$

where we employed (102) and (F-6). Actual numerical evaluation of (F-14), combined with V(0) from (F-3), gives incorrect results for the exceedance distribution function (F-1); thus the replacement of ρ with 0 in (F-5) is invalid. The explanation for this pitfall is the essential singularity of (F-1) at $\xi = -i/\omega$; a simpler illustration follows.

RESIDUE OF ESSENTIAL SINGULARITY

The function

$$\exp\left(\frac{1}{z}\right) = 1 + \frac{1}{z} + \frac{1}{2!\ z^2} + \ldots \tag{F-15}$$

has an essential singularity at z=0, with residue 1, as exemplified by this Laurent expansion. Now consider the function

$$f(z) = \exp\left(\frac{1}{z}\right)\ g(z)\ , \tag{F-16}$$

where g(z) is analytic at z=0. Then

$$f(z) = \left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots\right)\left(g(0) + g^{(1)}(0)\ z + \frac{1}{2!}g^{(2)}(0)\ z^2 + \ldots\right)\ . \tag{F-17}$$

The coefficient of 1/z in (F-17) is the residue of f(z) at z=0; namely

$$\text{Res} = g(0) + \frac{1}{2!}\ g^{(1)}(0) + \frac{1}{3!}\ \frac{1}{2!}\ g^{(2)}(0) + \ldots = \sum_{n=0}^{+\infty} \frac{g^{(n)}(0)}{n!(n+1)!}\ . \tag{F-18}$$

Thus the residue of f(z) at z=0 depends on the behavior of g(z) in a <u>neighborhood</u> of z=0, and not just the value g(0).

A couple of examples yield the following:

$$g(z) = (1-az)^{-1}\ , \quad \text{Res} = \frac{\exp(a)-1}{a}\ ;$$

$$g(z) = \exp(a^2 z)\ , \quad \text{Res} = \frac{I_1(2a)}{a}\ . \tag{F-19}$$

CORRECT APPROACH FOR $C(\rho)$

Reconsider the integral in (F-5) plus the scale factor $\rho^{1/2}$; making the substitution $z = \theta+iL$, where L is given in (F-8), there follows for this quantity

$$(2\lambda)^{1/2} \int_{-\pi+iL}^{\pi+iL} dz\ (1-2\omega\lambda\ e^{iz})^{-1}\ \left(1 - \frac{2\omega\lambda}{1+\omega}\ e^{iz}\right)^{-\frac{1}{2}} *$$

$$* \exp[i\ \frac{z}{2} + \frac{1}{2}\ e^{-iz} + 2\lambda u\ e^{iz}]\ . \tag{F-20}$$

The uppermost singularity of the integrand in the z-plane (within the $-\pi$, π strip) is a pole at $z_p = i\ \ell n(2\omega\lambda)$; however, the straight line contour in (F-20) remains above this pole because $\rho < 1/\omega$; see (F-8). Furthermore, the total integrand of (F-20) has a saddle point on the imaginary axis of the z-plane above the pole location z_p, because the integrand is infinite at the pole and at $z = 0 + i\infty$. Thus the straight line contour in (F-20) can be modified so as to pass through the saddle point, and yet remain above z_p. Finally, letting $\rho \rightarrow 0+$, then $L \rightarrow +\infty$, and (F-20) combined with (F-5) yields the exact result for the circular component

$$C(0) = \frac{r}{\pi\ 2^{1/2}\ (1+\omega)^{1/2}} \exp\left(-\ \frac{u+r^2}{\omega}\right) \int_{-\pi+i\infty}^{\pi+i\infty} dz\ (1-2\omega\lambda\ e^{iz})^{-1}\ *$$

$$*\left(1 - \frac{2\omega\lambda}{1+\omega}\ e^{iz}\right)^{-1/2} \exp\left[i\ \frac{z}{2} + \frac{1}{2}\ e^{-iz} + 2\lambda u\ e^{iz}\right], \tag{F-21}$$

where the two valleys of the integrand at $\pm\pi+i\infty$ are joined with a contour through the saddle point lying above the pole at $z_p = i\ \ell n(2\omega\lambda)$. Here $\omega = 1+2R$.

The other component of the exceedance distribution function, corresponding to (F-21), is given by (F-3) at $\rho = 0+$:

$$V(0) = \frac{2\,\omega^{1/2}}{\pi} \exp\left(-\frac{u+r^2}{\omega}\right) \int_0^{+\infty} dt\, (1+t^2)^{-1} *$$

$$*(1+\omega+t^2)^{-1/2} \exp\left(-\frac{u}{\omega} t^2 - \frac{r^2}{\omega t^2}\right). \qquad \text{(F-22)}$$

Thus for $u \geq 0$, (F-1) and figure F-1 yield exceedance distribution function

$$1 - P_h(u) = C(0) + V(0) = \text{(F-21)} + \text{(F-22)} . \qquad \text{(F-23)}$$

Computationally, (F-21) is not too attractive, because of the complex integrand and/or the need to determine the steepest descent paths to $\pm\pi+i\infty$ numerically. Accordingly, an alternative direct procedure for determining the exceedance distribution function of random variable h is now presented.

DIRECT EVALUATION OF EXCEEDANCE DISTRIBUTION FUNCTION

For $\gamma=0$, $N=1$, (12) and (3) yield the crosscorrelator output for the signal and noise model as

$$q = u_1 v_1 = [\mu_u + u_s(1) + u_d(1)]\,[\mu_v + v_s(1) + v_d(1)] . \qquad \text{(F-24)}$$

The normalized crosscorrelator output is then, from (49) and (97),

$$h = \frac{q}{(D_u D_v)^{1/2}} = (r_u + u_s' + u_d')\,(r_v + v_s' + v_d') \equiv xy , \qquad \text{(F-25)}$$

where x and y are joint Gaussian random variables with statistics

$$\overline{x} = r_u \ , \quad \overline{y} = r_v \ ,$$

$$\sigma_x^2 = 1 + R_u, \quad \sigma_y^2 = 1 + R_v, \quad \overline{(x-\overline{x})(y-\overline{y})} = \rho_s (R_u R_v)^{1/2} \ . \tag{F-26}$$

We now make the same assumptions as in (99); see also (56) et seq. Then (F-26) specializes to

$$\overline{x} = \overline{y} = r, \quad \sigma_x^2 = \sigma_y^2 = 1 + R \equiv \sigma^2, \quad \rho_{xy} = \frac{R}{1+R} \ . \tag{F-27}$$

The joint probability density function of x, y is then given by

$$p_2(x,y) = \left[2\pi\sigma^2\left(1-\rho_{xy}^2\right)^{1/2}\right]^{-1} \exp\left[-\frac{(x-r)^2 + (y-r)^2 - 2\rho_{xy}(x-r)(y-r)}{2\sigma^2(1-\rho_{xy}^2)}\right] . \tag{F-28}$$

We now have cumulative distribution function

$$P_h(u) = \iint\limits_{R_2+R_4} dx\, dy\, p_2(x,y) \quad \text{for } u \leq 0 \ , \tag{F-29}$$

and exceedance distribution function

$$1 - P_h(u) = \iint\limits_{R_1+R_3} dx\, dy\, p_2(x,y) \quad \text{for } u \geq 0 \ , \tag{F-30}$$

where regions R_1, R_2, R_3, R_4 are indicated in figure F-3.

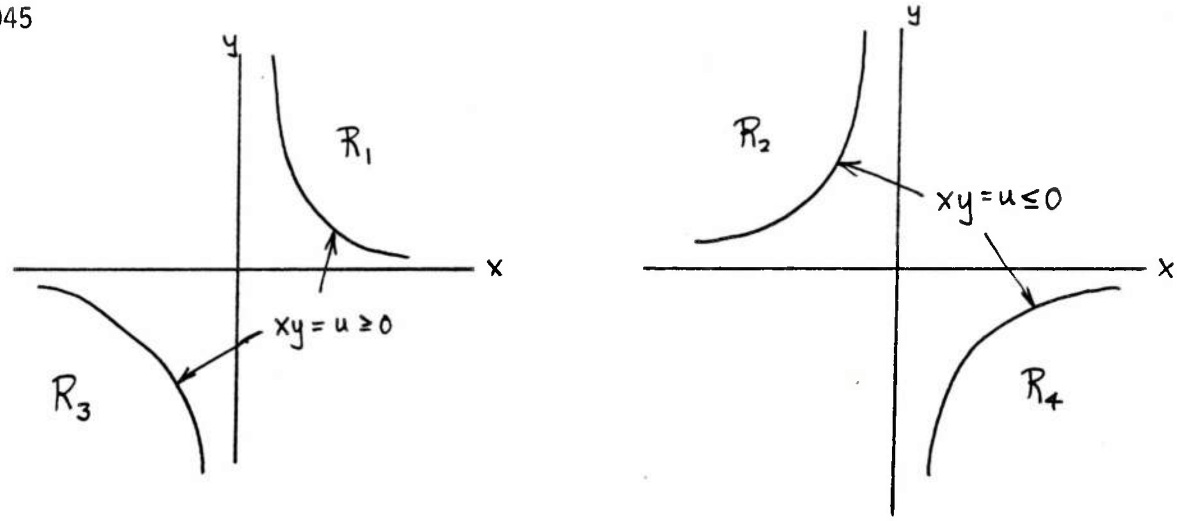

Figure F-3. Regions of Integration

If we now rotate axes according to

$$s = \frac{x+y}{2}\ , \quad t = \frac{x-y}{2} \tag{F-31}$$

and employ (84) and (F-27), there follows (after scale changes of the variables)

$$P_h(u) = 2\int_{-\infty}^{+\infty} dv\ \phi\left(v - r(2/\omega)^{1/2}\right)\Phi\left(-(\omega v^2 - 2u)^{1/2}\right) \quad \text{for } u \le 0\ , \tag{F-32}$$

and

$$1 - P_h(u) = 2\int_0^{+\infty} dv\ \phi(v)\left[\Phi\left(\frac{r - \sqrt{u+v^2/2}}{\sqrt{\omega/2}}\right) + \Phi\left(\frac{-r - \sqrt{u+v^2/2}}{\sqrt{\omega/2}}\right)\right] \text{for } u \ge 0\ . \tag{F-33}$$

Here $\omega = 1+2R$. These real integrals are very useful for the evaluation of the distributions of h when $\gamma=0$, $N=1$. In fact, (F-33) is preferred over (F-21)-(F-23); but (106) is preferred over (F-32) since Φ need not be evaluated in (106). This is in fact the procedure utilized here to obtain numerical results for this case of $\gamma=0$, $N=1$.

APPENDIX G. PROGRAM FOR EVALUATION OF OPERATING CHARACTERISTICS FOR γ=0

The comments in appendices C and D are relevant here also. The characteristic function used as the starting point is given by (100). It was observed under (103) that $|f_h(\xi)|$ for (100) is monotonically decreasing for all $\xi \geq 0$; thus the choice of truncation value L is simplified; see appendix D comments. A table for sampling increment Δ_0 (when R=0) follows.

N	Δ_0 for r=1	Δ_0 for r=2
3	.07	.05
4	.07	.05
8	.05	.03
16	.04	.02
32	.03	.02
64	.025	.015
128	.020	.010
256	.012	.007

Table G-1. Values of Δ_0 for γ=0

```
10 !  GAMMA = 0          NO SAMPLE MEAN REMOVAL
20    Nc=32                       !  N, Number of terms added
30    Rs=1                        !  r, Normalized mean
40    Delta0=.03                  !  Initial delta
50    Bs=2*PI/Delta0*.375         !  Bias b (depends on r)
60    Mf=2^10                     !  Size of FFT
70    OUTPUT 0;"GAMMA = 0";"     N =";Nc;"     r =";Rs
80    OUTPUT 0;""
90    DATA -4,-3,-2.5,-2,-1.5,-1,-.5,0,.5,1,1.5,2
100   READ Ns(*)                  !  SNR R=2^n
110   OUTPUT 0;Ns(*);
120   DATA 1,2,2,2,2,2,2,2,2,4,4,4,8
130   READ Idelta(*)
140   MAT Delta=(Delta0)/Idelta
150   OUTPUT 0;Delta(*);
160   DATA 1E-10,1E-9,1E-8,1E-7,1E-6,1E-5,1E-4,.001,.01,.1,.5,.9,.99,.999
170   READ Sc(*)
180   DIM Ns(1:12),Idelta(0:12),Delta(0:12),Sc(1:14)
190   DIM X(0:8191),Y(0:8191)
200   FOR I=1 TO 14
210   Sc(I)=FNInvphi(Sc(I))
220   NEXT I
```

```
230    S=Sc(1)
240    B=Sc(14)
250    Scale=(B-S)/(0-S)
260    X1=30
270    X2=170
280    Y1=35
290    Y2=Y1+(X2-X1)*Scale
300    PLOTTER IS "9872A"
310    LIMIT X1,X2,Y1,Y2
320    OUTPUT 705;"VS3"
330    SCALE S,0,S,B
340    FOR I=1 TO 14
350    MOVE S,Sc(I)
360    DRAW 0,Sc(I)
370    NEXT I
380    FOR I=1 TO 11
390    MOVE Sc(I),S
400    DRAW Sc(I),B
410    NEXT I
420    MOVE S,S
430    DRAW 0,0
440    PENUP
450    M1=Mf-1
460    N2=Nc/2
470    Rsn=Nc*Rs*Rs                  !  N r^2
480    FOR In=0 TO 12
490    IF In>0 THEN 520
500    Rc=0
510    GOTO 530
520    Rc=2^Ns(In)                   !  SNR R=2^n
530    OUTPUT 0;"R =";Rc,"     Delta =";Delta(In)
540    ASSIGN #1 TO "ABSCIS"         !  Temporary storage
550    Delta=Delta(In)               !  for false alarm probability
560    R2=Rc*2
570    R21=R2+1
580    Mux=Nc*Rc+Rsn                 !  Mean of random  variable h
590    Muy=Mux+Bs                    !  Mean of shifted variable y
600    REDIM X(0:M1),Y(0:M1)
610    MAT X=ZER
620    MAT Y=ZER
630    X(0)=0
640    Y(0)=.5*Delta*Muy
650    Ls=0
660    Ls=Ls+1
670    Xi=Delta*Ls                   !  Argument xi of char. fn.
680    Ei=Xi*R21                                       !  Calculation
690    CALL Log(1+Xi*Ei,-Xi*R2,Ai,Bi)                  !  of
700    CALL Div(0,Xi*Rsn,1,-Ei,Ci,Di)                  !  characteristic
710    CALL Exp(Ci-N2*Ai,Di+Xi*Bs-N2*Bi,Fyr,Fyi)       !  function
720    Ms=Ls MOD Mf                                    !  fy(xi)
730    Ar=Fyr/Ls
740    Ai=Fyi/Ls
750    X(Ms)=X(Ms)+Ar
760    Y(Ms)=Y(Ms)+Ai
770    Magsq=Ar*Ar+Ai*Ai
780    IF Magsq>1E-24 THEN 660
790    OUTPUT 0;"Xi =";Xi;"     Mag =";SQR(Magsq)
800    CALL Fft13z(Mf,X(*),Y(*))
```

```
810   FOR Ms=0 TO M1
820   T=Y(Ms)/PI-Ms/Mf
830   X(Ms)=.5-T                  !  Cumulative distribution function
840   Y(Ms)=.5+T                  !  Exceedance distribution function
850   NEXT Ms
860   OUTPUT 0;Y(0);Y(1);Y(M1-1);Y(M1)
870   PLOTTER IS "GRAPHICS"
880   GRAPHICS
890   SCALE 0,Mf,-14,0
900   LINE TYPE 3
910   GRID Mf/8,1
920   PENUP
930   LINE TYPE 1
940   FOR Ms=0 TO M1
950   Pr=Y(Ms)
960   IF Pr>=1E-12 THEN Y=LGT(Pr)
970   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
980   IF ABS(Pr)<1E-12 THEN Y=-12
990   PLOT Ms,Y
1000  NEXT Ms
1010  PENUP
1020  FOR Ms=0 TO M1
1030  Pr=X(Ms)
1040  IF Pr>=1E-12 THEN Y=LGT(Pr)
1050  IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
1060  IF ABS(Pr)<1E-12 THEN Y=-12
1070  PLOT Ms,Y
1080  NEXT Ms
1090  PENUP
1100  DUMP GRAPHICS
1110  OUTPUT 0;""
1120  IF In>0 THEN 1270
1130  FOR Ms=0 TO M1
1140  IF Y(Ms)<.7 THEN 1160
1150  NEXT Ms
1160  M2=Ms-1
1170  FOR Ms=M2 TO M1
1180  IF Y(Ms)<=0 THEN 1200
1190  NEXT Ms
1200  M3=Ms-1
1210  REDIM X(M2:M3)
1220  FOR Ms=M2 TO M3
1230  X(Ms)=FNInvphi(Y(Ms))
1240  NEXT Ms
1250  PRINT #1;X(*)               !  Store false alarm probability
1260  GOTO 1460
1270  REDIM X(M2:M3)
1280  READ #1;X(*)                !  Read in false alarm probability
1290  Id=Idelta(In)
1300  J2=INT(M2/Id)
1310  J3=INT(M3/Id)+1
1320  FOR J=J2 TO J3
1330  Y(J)=FNInvphi(Y(J))
1340  NEXT J
```

```
1350  PLOTTER IS "9872A"
1360  LIMIT X1,X2,Y1,Y2
1370  OUTPUT 705;"VS3"
1380  SCALE S,0,S,B
1390  FOR J=J2 TO J3
1400  T=J*Id
1410  IF T<M2 THEN 1440
1420  IF T>M3 THEN 1450
1430  PLOT X(T),Y(J)
1440  NEXT J
1450  PENUP
1460  NEXT In
1470  END
1480  !
1490  SUB Div(X1,Y1,X2,Y2,A,B)            !  DIV(Z)
1540  !
1550  SUB Exp(X,Y,A,B)                    !  EXP(Z)
1600  !
1610  SUB Log(X,Y,A,B)                    !  PRINCIPAL LOG(Z)
1690  !
1700  DEF FNInvphi(X)             !  INVPHI(X) via AMS 55, 26.2.23
1830  !
1840  SUB Fft13z(N,X(*),Y(*))     !  N <= 2^13, N=2^INTEGER, 0 SUBSCRIPT
```

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